DOCUMENT RESUME

ED 173 100 SE 027 907

AUTHOR	Allen, Frank B.; And Others
TITLE	Mathematics for High School, First Course in Algebra, Part 2. Preliminary Edition.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	59
NOTE	239p.; For related documents, see ED 135 617-618; Contains occasional light and broken type
EDRS PRICE	MF01/PC10 Plus Postage.
DESCRIPTORS	*Algebra; Curriculum; *Instruction; Mathematics Education; *Number Concepts; Secondary Education; *Secondary School Mathematics; *Textbooks
IDENTIFIERS	*Polynomials; *School Mathematics Study Group

ABSTRACT

This is part two of a three-part SMSG algebra text for high school students. The principle objective of the text is to help the student develop an understanding and appreciation of some of the algebraic structure exhibited by the real number system, and the use of this structure as a basis for the techniques of algebra. Chapter topics include addition and multiplication of real numbers, subtraction and division of real numbers, factors, exponents, radicals, and polynomials and rational expressions. Moderate revisions are contained in a later edition. (MP)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED173100

SCHOOL
MATHEMATICS
STUDY GROUP

U.S. DEPARTMENT OF HEALTH,
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY
SMSG

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

MATHEMATICS FOR HIGH SCHOOL

FIRST COURSE IN ALGEBRA (Part 2)

(preliminary edition)

2

SE 027 907

MATHEMATICS FOR HIGH SCHOOL

FIRST COURSE IN ALGEBRA (Part 2)

(preliminary edition)

Prepared under the supervision of the Panel on Sample Textbooks

of the School Mathematics Study Group:

Frank B. Allen	Lyons Township High School
Edwin C. Douglas	Taft School
Donald E. Richmond	Williams College
Charles E. Rickart	Yale University
Henry Swain	New Trier Township High School
Robert J. Walker	Cornell University

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Copyright 1959 *by Yale University.*

PHOTOLITHOPRINTED BY CUSHING - MALLOY, INC.
ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA

FIRST COURSE IN ALGEBRA

(Part II)

Table of Contents

Page

Chapter 6 Addition and Multiplication of Real Numbers 141

Chapter 7 Subtraction and Division of Real Numbers 196

10, 11 Chapter 8 Factors, Exponents, Radicals 254

12 Chapter 9 Polynomials and Rational Expressions 326

Chapter 6

Addition and Multiplication of Real Numbers

6 - 1. Addition of real numbers. Ever since first grade you have been adding numbers, the numbers of arithmetic. Now we are dealing with a larger set of numbers, the real numbers. Our long experience with adding numbers of arithmetic should quickly give us a clue as to how we add any real numbers.

Let us consider the profits and losses of an imaginary ice cream vender during his 10 days in business. Instead of using black and red ink, let us use positive numbers to represent profit and negative numbers for losses. Then let us write two columns, one for the description of his business activities and the other for the arithmetic of computing his net income over two-day periods.

Business	Arithmetic
Mon.: Profit of \$7 Tues.: Profit of \$5	$7 + 5 = 12$, net income.
Wed.: Profit of \$6 Thurs.: Loss of \$4 (tire trouble)	$6 + (-4) = 2$, net income.
Fri.: Loss of \$7 (another tire) Sat.: Profit of \$4	$(-7) + 4 = -3$, net income.
Sun.: Day of rest. Mon.: Loss of \$3 (cold day)	$0 + (-3) = -3$, net income.

Tues.: Loss of \$4 (colder)	$(-4) + (-6) = -10$, net income.
Wed.: Loss of \$6 (gave up)	

These accounts illustrate every possible sum of real numbers: the sum of two positives, of a positive and a negative, of two negatives, and a sum involving 0. Although at first you might prefer to think of positives and negatives as gains and losses, soon you will begin to add real numbers themselves, without reference to such an illustration. By the way, how much did the vendor gain or lose during his 10 days in business?

With this illustration of gains and losses as guide, you should decide how to complete the following:

$\frac{1}{2} + 2 =$ $(-4) + 2 =$

$0 + \frac{4}{3} =$ $(-\frac{7}{5}) + 0 =$

$3 + (-1) =$ $(-4) + (-\frac{1}{2}) =$

In case you were in any doubt, could you resolve the question? How did you think about it?

Exercises 6 - 1a.

1. If the gains and losses example has not made clear how we are going to use real numbers in addition, you should work out some of the problems below. If you have a clear idea about addition, you should go on to problem 2.
   (a) A football team lost 6 yards on the first play and gained 8 yards the second play. What was the net yardage on the

two plays?

(b) John paid Jim the 60¢ he owed, but John collected the 50¢ Al owed him. What is the net result of John's two transactions?

(c) Suppose we agree that years A.D. are represented by positive numbers and years B.C. by negative numbers.

  (1) If A was born in 450 B.C. and B was born 100 years later, in what year was B born?

  (2) If C was born in 150 B.C. and D was born 200 years earlier, in what year was D born?

  (3) If E was born in 350 B.C. and F was 150 years earlier, when was F born?

  (4) If G was born in 240 A.D. and H was born 220 years earlier, when was H born?

(d) If a thermometer registers -15 degrees and the temperature rises 10 degrees, what does the thermometer then register? What if the temperature had risen 30 degrees instead?

(e) A certain stock market price gained two points one day, lost five points the next day, lost two points the third day. What was its net change?

(f) Miss Jones lost 6 pounds during the first week of her dieting, lost 3 pounds the second week, gained 4 pounds the third week, gained 5 pounds the last week. What was her net gain or loss?

(g) The national debt ceiling was \$300 billion, and Congress voted to increase the debt ceiling by \$60 billion. Then what was the ceiling?

(h) Represent North latitude with positive numbers and South latitude with negative.

(1) City A is at 41° N and B is 12° south of A. Where is B?

(2) City C is at 30° S and D is 10° north of C. Where is D?

(3) City E is at 15° S and F is 40° north of E. Where is F?

(4) City G is at 15° N and H is 40° south of G. Where is H?

2. Perform the indicated operations on real numbers:

(a) $(4 + (-6)) + (-4)$

(b) $4 + ((-6) + (-4))$

(c) $-(4 + (-6))$

(d) $3 + ((-2) + 2)$

(e) $2 + (0 + (-2))$

(f) $((-5.4) + 3.6) + 1.8$

(g) $((-3) + 0) + (-2.5)$

(h) $|-2| + (-2)$

(i) $|3 + (-5)| + 2$

(j) $((-7) + (-4)) + (5 + (-9))$

(k) $((-19) + 37) + (-23)$

(ℓ) $(-\frac{7}{5}) + \frac{6}{5}$

(m) $(-3.2) + ((-4.3) + 3.0)$

(n) $(-\frac{1}{3}) + (2 + (-\frac{2}{3}))$

(o) $(-3) + (|-3| + 5)$

(p) $(6 + (-7)) + |-3|$

(q) $4 + ((-5) + (-7))$

(r) $(0 + |-3|) + ||-4| + (-4)|$

3. Tell in your own words what you do to the two given numbers to find their sum:

(a) 7 + 10	(d) (-10) + (-7)	(g) (-7) + 10
(b) 7 + (-10)	(e) 10 + 7	(h) (-10) + 7
(c) 10 + (-7)	(f) (-7) + (-10)	

4. In which parts of problem 3 did you do the addition just as you added numbers in arithmetic?
5. When one number was positive and the other was negative, how did you know whether the sum was positive or negative?
6. What could you always say about the sum when both numbers were negative?
7. Would talking about the absolute values of these numbers be a convenience in talking about how you add them? How would it help? As an example, write the sum (-10) + (-7) using absolute value signs.
8. Can the sum of two real numbers ever be less than either one of them?
9. If one number is positive and the other is negative, what do you do with the absolute value of each of the numbers to get the absolute value of their sum? As an example, write the sum (-10) + 7 using absolute value signs.

The preceding exercises have probably suggested that it will not be possible to define addition of real numbers in one simple sentence. If, however, you have thought carefully about the exercises, you will probably agree readily that all that we *do* have to say about adding real numbers is summarized in the following definition.

Definition: If a and b are real numbers, we define their *sum* as follows:

I. If a and b are both positive or zero, the *sum* is their sum as in arithmetic.

II. If a and b are both negative, the sum is the *negative* of the *sum* of their absolute values.

III. If one of a and b is positive or zero and the other is negative, the absolute value of their sum is the *difference** of their absolute values. Then, if the sum is not 0, it is positive or negative according as the positive or the negative number has the greater absolute value.

Examples: Both positive: $9 + 20 = 29$

Both negative: $(-9) + (-20) = -(9 + 20) = -29$

One positive and one negative: $(+9) + (-20) = -(20 - 9) = -11$

Negative because $|-20| > |9|$.

$(-9) + (+20) = (20 - 9) = 11$

Positive because $|20| > |-9|$

$(-9) + 9 = 0$

Zero because $|-9| = |9|$

---

*Remember that the difference of two non-negative numbers is the larger minus the smaller if they are different, and is zero if they are the same.

By this time you have used the symbols of algebra enough so that you may find it convenient to summarize the above in the following form (for the case $|a| \geq |b|$):

If a and b are real numbers

and

if both are positive or zero, their sum is $a + b$;

if both are negative, their sum is $-(|a| + |b|)$;

if one is positive or zero and one is negative,

$$\text{their sum is } \begin{cases} (|a| - |b|) & \text{if } a \text{ is positive,} \\ -(|a| - |b|) & \text{is } a \text{ is negative.} \end{cases}$$

Try to write this summary for the case $|b| \geq |a|$.

While we shall find this definition useful as a basis for later ideas about addition, there is no reason why you should not use whatever viewpoint is most comfortable for you when you are interested just in finding a sum of two real numbers. To be sure, however, that you really understand the formal definition, try the following exercises.

Exercises 6 - 1b.

1. In each of the following find the sum, first according to the definition, then by whatever method you find most convenient:

(a) $3 + (-5)$

(b) $(-5) + (-11)$

(c) $0 + (-15)$

(d) $\sqrt{2} + (-\sqrt{2})$

(e) $18 + (-14)$

(f) $(-\pi) + \pi$

(g) $(-\frac{2}{3}) + \frac{1}{2}$

(h) $(-35) + (-65)$

(i) $12 + 7$

(j) $(-6) + 10$

(k) $1 + (-\frac{3}{2})$

(l) $200 + (-201)$

2. Try to write out parts I and II of the definition of addition in terms of the symbols $<$, $\leq$, $|\ |$. Is this a more compact form?

3. Given the set $K = \{0, -1, 2, -\frac{1}{2}\}$, find the set S of all sums of pairs of elements of K. Is K closed under the operation of addition? (Review the meaning of closure of a set under an operation before you justify your answer.)

4. Is the set of all real numbers closed under the operation of addition?

5. Is the set of all negative real numbers closed under addition? Justify your answer.

6. In the course of a week the variations in mean temperature from the seasonal normal of 71 were -7, 2, -3, 0, 9, 12, -6. What were the mean temperatures each day. What is the sum of the variations?

7. Which of the following are true sentences?

(a) $(-4) + 0 = 4$

(b) $-(|-1.5| - |0|) = -1.5$

(c) $(-3) + 5 = 5 + (-3)$

(d) $\left(4 + (-6)\right) + 6 = 4 + \left((-6) + 6\right)$

(e) $(-5) + \big(-(-5)\big) = -10$

(f) $(-7) + \big((-5) + (-3)\big) = \big((-7) + (-5)\big) + 3$

(g) $-\big(6 + (-2)\big) = (-6) + (-2)$

(h) $(-7) + (-9) = -(7 + 9)$

(i) $(-3) + 7 = -\big(3 + (-7)\big)$

8. For the following open sentences try to find a real number which will make the sentence true:

(a) $x + 2 = 7$

(b) $3 + y = -7$

(c) $a + 5 = 0$

(d) $b + (-7) = 3$

(e) $(-\frac{5}{6}) + x = -\frac{5}{6}$

(f) $c + (-3) = -7$

(g) $y + \frac{2}{3} = -\frac{5}{6}$

(h) $\frac{1}{2}x + (-4) = 6$

(i) $\big(y + (-2)\big) + 2 = 3$

(j) $(3 + x) + (-3) = -1$

9. What is the sum of:

(a) 4 and -4?

(b) -2 and $-(-2)$?

(c) $-\frac{5}{2}$ and $\frac{5}{2}$?

(d) Formulate a general rule for the sum of a number and its opposite.

10. If it is known that $x + (-2)$ and -5 are the same number, is it then true that $\big(x + (-2)\big) + 2$ and $(-5) + 2$ are the same number? Why?

11. Translate the following English sentences into open sentences. For example: Bill spent 60¢ on Tuesday and earned 40¢ on Wednesday. He couldn't remember what happened on Monday, but he had 30¢ left on Wednesday night. What did he spend or earn on Monday?

If Bill spent or earned x cents on Monday, then

$$x + (-60) + 40 = 30.$$

This can be written

$$x + (-20) = 30.$$

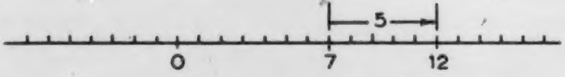
(a) A stock price lost 3 points in the morning, and by evening it was down a total of 7 points. What happened to the stock during the afternoon?

(b) After borrowing \$2 more from his brother and paying \$4 back, Jim found that he still owed his brother \$4. What was his original debt to his brother?

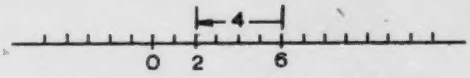
(c) At the end of the year the assets of a business firm are \$101,343 and the liabilities are \$113,509. What is the balance?

(d) From a submarine submerged 215 feet below the surface a rocket is fired which rises 3,000 feet. How far above sea level does the rocket go?

In all our work with numbers in earlier chapters we have made use of the number line in representing facts about numbers. Let us now go back to the business venture of the ice cream vendor and illustrate his net incomes over successive two day periods:

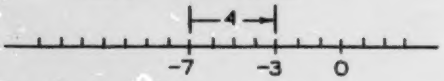

Mon. and Tues.

$7 + 5 = 12$


Wed. and Thurs.

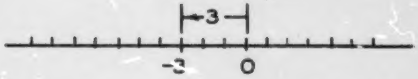
$6 + (-4) = 2$

Fri. and **Sat.**



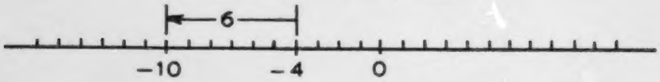
$(-7) + 4 = -3$

Sun. and **Mon.**



$0 + (-3) = -3$

Tues. and **Wed.**



$(-4) + (-6) = -10$

You should recall, of course, that to add a positive number we move to the right on the number line. Now it should also be clear that to add a negative number we move to the left on the number line. In what direction do we move to add 0 on the number line? This can be summarized:

> To determine the sum $a + b$ on the real number line, start at a and move $|b|$
>
> to the right, if b is positive,
> to the left, if b is negative.

Exercises 6 - 1c

1. Which of the following are true sentences?
   - (a) $|-3| + 4 \neq -7$
   - (b) $(-2) + 5 < 6 + (-4)$
   - (c) $(-1) + 0 < |-1|$
   - (d) $6 + (-7) > |6 + (-7)|$
   - (e) $|-3| + |-2| \geq |(-3) + (-2)|$
   - (f) $(5 + |-2|) + (-7) \neq 0$

(g) $(-3) + (-5) \ngtr |-3| + |-5|$

(h) $3(2) + (-4) < |-4| + (-6)$

2. Starting at the point with coordinate $-\frac{4}{5}$, move right 3, left $\frac{6}{5}$, left 4, right 2. What is the coordinate of the stopping point?

3. Find the greatest of the numbers: $(-2) + (-3)$, $-\big((-2) + (-3)\big)$, $(-2) + 3$, $2 + (-3)$, $|(-2) + 3| + |2 + (-3)|$.

6 - 2. Properties of Addition. We were careful to describe and list the properties of addition when we dealt with the numbers of arithmetic. Now that we have decided how to add real numbers, we want to verify that these properties of addition hold true for the real numbers generally.

We know that our definition of addition includes the usual addition of numbers of arithmetic, but we also want to be able to add as simply as we could before. Can we still add real numbers in any order and group them in any way to suit our convenience? In other words, do the commutative and associative properties of addition still hold true? If we are able to satisfy ourselves that these properties do carry over to the real numbers, then we are assured that the structure of numbers is maintained as we move from the numbers of arithmetic to the real numbers. Similar questions about multiplication will come up later.

Consider the following true sentences:

$(-5) + (-2) = (-2) + (-5)$, $7 + (-4) = (-4) + 7$, $3 + (-6) = (-6) + 3$. What do we do to form the sum $(-5) + (-2)$?

(Say it in words.) What do we do to form the sum $(-2) + (-5)$? Why are these two the same number?

Say what you do to form the sum $7 + (-4)$; the sum $(-4) + 7$. Why in this case are these two the same number?

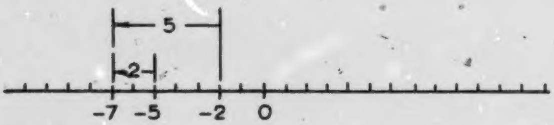
Do the above sentences cover every possible case of addition of real numbers? If not, supply examples of the missing cases.

By now you see the point: The sum of any two real numbers is the same for either order of addition. This is the

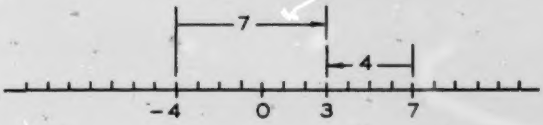
Commutative Property of Addition: For any two real numbers a and b,

$$a + b = b + a.$$

Now it should seem clear that the sum of two numbers is defined without regard to which of the numbers is taken first. But on the number line, where addition seems to depend on which number is taken first, the commutative property is not so obvious.



$(-5) + (-2) = (-2) + (-5)$



$7 + (-4) = (-4) + 7$

Next, compute the following pairs of sums:

$$\big(7 + (-9)\big) + 3, \qquad 7 + \big((-9) + 3\big);$$
$$\big(8 + (-5)\big) + 2, \qquad 8 + \big((-5) + 2\big);$$
$$(4 + 5) + (-6), \qquad 4 + \big(5 + (-6)\big).$$

What do you observe about the results?

We could list many more examples. Do you think the same results would always hold? Again you should see the point. This is the

Associative Property of Addition: For any real numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

Of course, if the associative and commutative properties hold true in several instances it is not a proof that they will hold true in every instance. A complete proof of the properties can be given by applying the precise definition of addition of real numbers to every possible case of the properties. They are long proofs, especially of the associative property, because there are many cases. We shall not take the time to give the proofs, but perhaps you may want to try the one for the commutative property.

The associative property assures us that in a sum of three real numbers it doesn't matter which adjacent pair we add first; it is customary to drop the parentheses and leave such sums in an unspecified form, such as $4 + (-1) + 3$.

Another property of addition, which is new for real numbers and one that we will find useful, is obtained from the definition of addition. For example, the definition tells us that $4 + (-4) = 0$; that $(-4) + \big(-(-4)\big) = 0$. In general, the sum of a number and its opposite is 0. We state this as the

Addition Property of Opposites: For every real number a,

$$a + (-a) = 0.$$

One more property that stems directly from the definition is the

Addition Property of 0: For every real number a,

$$a + 0 = a.$$

Make up several examples to illustrate this and the preceding property.

To complete our list of basic properties of addition, consider the examples:

Since $4 + (-5) = -1$, then $\left(4 + (-5)\right) + 5 = (-1) + 5$. (Verify this.)

Since $7 = 15 + (-8)$, then $7 + (-7) = \left(15 + (-8)\right) + (-7)$. (Verify this.) Notice that since "$4 + (-5)$" and "(-1)" are names for the same number, we may add 5 to this number and obtain "$\left(4 + (-5)\right) + 5$" and "$(-1) + 5$" as names for the same new number. Do you see the point? We call this the

Addition Property of Equality. For any real numbers a, b, c,

$$\text{if } a = b, \text{ then } a + c = b + c.$$

Let us make use of these properties in some examples.

Example 1. Compute the sum: $(-5) + 7 + 9 + (-12) + 5$.

The commutative and associative properties allow us to reorder and group real numbers in any manner for addition. Thus we write

$$\begin{aligned}(-5) + 7 + 9 + (-12) + 5 &= \big((-5) + 5\big) + \big((7 + 9) + (-12)\big) \\ &= 0 + \big(16 + (-12)\big) \\ &= 0 + 4 \\ &= 4.\end{aligned}$$

Notice that it is useful to group opposites, if possible, because their sum is always 0. In fact, we could write

$$\begin{aligned}16 + (-12) &= (4 + 12) + (-12) \\ &= 4 + \big(12 + (-12)\big) \qquad \text{(Why?)} \\ &= 4 + 0 \qquad \text{(Why?)} \\ &= 4.\end{aligned}$$

Example 2. Determine the truth set of the open sentence

$$x + \frac{3}{5} = -2.$$

Can you guess numbers which make this sentence true? If you don't see it easily, could you use properties of addition to help? Let us see. If there is a number x making the sentence true (if the truth set is not empty), then

$x + \frac{3}{5}$ and -2 are the same number.

Let us add $-\frac{3}{5}$ to this number; then by the addition property of equality we have

$$\left(x + \frac{3}{5}\right) + \left(-\frac{3}{5}\right) = (-2) + \left(-\frac{3}{5}\right). \qquad \text{(Why did we add } -\tfrac{3}{5}\text{?)}$$

Continuing, we have

$$x + \left(\frac{3}{5} + \left(-\frac{3}{5}\right)\right) = (-2) + \left(-\frac{3}{5}\right), \quad \text{(Why?)}$$

$$x + 0 = -\frac{13}{5}, \quad \text{(Why?)}$$

$$x = -\frac{13}{5}.$$

Thus, we arrive at the new open sentence $x = -\frac{13}{5}$. Now if a number x makes the original sentence true, it also makes this new sentence true. Of this we are certain because we applied properties which hold true for all real numbers. Does $-\frac{13}{5}$ make the original sentence true? Yes, because $(-\frac{13}{5}) + \frac{3}{5} = -2$.

Here we have hit upon a very important idea. We have shown that if there is a number x making the original sentence true, then the only number x can be is $-\frac{13}{5}$. The minute we check and find that $-\frac{13}{5}$ does make the sentence true, we have found the one and only number which belongs to the truth set.

Example 3. Determine the truth set of

$$5 + \frac{3}{2} = x + \left(-\frac{1}{2}\right).$$

If there is a number x such that the sentence is true, then

$$\left(5 + \frac{3}{2}\right) + \frac{1}{2} = \left(x + \left(-\frac{1}{2}\right)\right) + \frac{1}{2},$$

$$5 + 2 = x + 0,$$

$$7 = x.$$

If $x = 7$, then

the left side is $5 + \frac{3}{2} = \frac{10}{2} + \frac{3}{2} = \frac{13}{2}$;

the right side is $x + (-\frac{1}{2}) = 7 + (-\frac{1}{2}) = \frac{14}{2} + (-\frac{1}{2}) = \frac{13}{2}$.

Hence, the truth set is $\{7\}$.

Examples 2 and 3 illustrate a technique that we will use often in mathematics. We apply general properties of numbers to open sentences to obtain simpler open sentences, and then find the truth sets of the simpler sentences. These new truth sets often are the same as those of the original sentences, as was the case in Examples 2 and 3.

Exercises 6 - 2a.

1. Tell why each of the following sentences is true. (Specify associative, commutative properties, addition property of 0, or addition property of opposites.)
   - (a) $3 + \left((-3) + 4\right) = 0 + 4$
   - (b) $\left(5 + (-3)\right) + 7 = \left((-3) + 5\right) + 7$
   - (c) $\left(7 + (-7)\right) + 6 = 6$
   - (d) $|-1| + |-3| + (-3) = 1$
   - (e) $(-2) + \left(3 + (-4)\right) = \left((-2) + 3\right) + (-4)$
   - (f) $(-|-5|) + 6 = 6 + (-5)$
2. Regroup and reorder the following to obtain the sums in an easy way:
   - (a) $(-\frac{1}{2}) + 7 + (-2) + (-\frac{3}{2}) + 2$
   - (b) $\frac{5}{3} + (-3) + 6 + \frac{1}{3} + (-2)$
   - (c) $253 + (-67) + (-82) + (-133)$
   - (d) $(-3) + 8 + 11 + (-5) + (-3) + 12 + (-4)$
   - (e) $\frac{2}{3} + \frac{3}{2} + (-\frac{5}{3}) + (-\frac{1}{2}) + |-2|$

(f) $\frac{25}{4} + (-3) + |-\frac{3}{4}| + (-7)$

(g) $|-\frac{3}{2}| + \frac{5}{2} + (-7) + |-4|$

(h) $(x + 2) + (-x) + (-3)$

(i) $w + (w + 2) + (-w) + 1 + (-3)$

3. Find the truth set of each of the open sentences:
(Use the technique of Examples 2 and 3 wherever possible.)

(a) $(-6) + 7 = (-8) + x$

(b) $(-1) + 2 + (-3) = 4 + x + (-5)$

(c) $(x + 2) + x = (-3) + x$

(d) $(-2) + x + (-3) = x + (-\frac{5}{2})$

(e) $|x| + (-3) = |-\frac{2}{3}| + (-\frac{3}{2})$

(f) $(-\frac{3}{16}) + |x| = \frac{3}{4} + (-\frac{3}{8})$

(g) $x + (-3) = |-4| + (-3)$

(h) $(-\frac{4}{3}) + (x + \frac{1}{2}) = x + (x + \frac{1}{2})$

We have leaned heavily on the fact that the sum of a number and its opposite is 0. Let us examine this idea more carefully.

Two numbers whose sum is 0 are related in a very special way. For example, what number when added to 3 yields the sum 0? What number when added to -4 yields 0? In general, if x and y are real numbers such that $x + y = 0$, we say that y is an *additive inverse* of x.

A quick inspection of the sentence $x + y = 0$ shows that if $y = -x$, then the sentence is true. In other words, one additive inverse of x is its opposite $-x$.

Are there any other additive inverses of a number besides its opposite? How do we know there is no other strange inverse z different from $-x$? We can quickly settle this question by applying the addition property of equality: If z is a number such that

$$x + z = 0,$$

then

$$(-x) + (x + z) = (-x) + 0, \qquad \text{(Why?)}$$

and

$$\big((-x) + x\big) + z = -x, \qquad \text{(Why?)}$$

and

$$0 + z = -x, \qquad \text{(Why?)}$$

and

$$z = -x. \qquad \text{(Why?)}$$

This result verifies what we suspected from the beginning: The only additive inverse of a real number x is its opposite, $-x$.

The preceding paragraph was probably a new experience for you. Instead of showing that a specific number, say -4, has only one additive inverse 4, we showed that each real number x has only one additive inverse $-x$. We did this by applying properties which are true for each real number. The conclusion is important, although we suspected it all along; the technique is important, because it enables us to show general properties for all real numbers.

Let us look at another example of this technique of showing a general property of numbers. We call this a proof of the property. We know that

$$-(3 + 5) = (-3) + (-5),$$
$$-\big(4 + (-2)\big) = (-4) + 2,$$
$$-\big((-1) + (-4)\big) = 1 + 4,$$

are true sentences according to the definition of addition. (Verify this.) From these examples we might guess:

For any real numbers a and b it is true that $-(a + b) = (-a) + (-b)$.

Another way of saying this is: The opposite of the sum of two numbers is the sum of the opposites of the numbers. What we have here is a general statement relating the operations of addition and taking the opposite. Again it is easy to show this result for specific numbers, but is the result true for all real numbers? We reason as follows: If we want to show that the opposite of $a + b$ is $(-a) + (-b)$, we can do this simply by showing that the sum of $a + b$ and $(-a) + (-b)$ is 0. Why? It is natural then to write down this sum and show that it is 0.

$$\begin{aligned}
(a + b) + \big((-a) + (-b)\big) &= a + b + (-a) + (-b) && \text{Why?}\\
&= \big(a + (-a)\big) + \big(b + (-b)\big) && \text{Why?}\\
&= 0 + 0 && \text{Why?}\\
&= 0 . && \text{Why?}
\end{aligned}$$

If you have given reasons for each step in the above proof, you see that we have given a general result for any real numbers because we applied only properties which are true for any real numbers. This distinction between results for specific numbers and results for any numbers is a distinction between arithmetic and algebra.

Exercises 6 - 2b.

1. Find a number for which each sentence is true:

   (a) $3 + x = 0$
   (b) $(-2) + a = 0$
   (c) $3 + 5 + y = 0$
   (d) $x + (-\frac{1}{2}) = 0$
   (e) $|-4| + 3 + (-4) + c = 0$
   (f) $\left(-\left(-\frac{2}{3}\right)\right) + y = 0$
   (g) $\left(-\left(2 + \frac{1}{3}\right)\right) + a = 0$
   (h) $2 + x + (-5) = 0$
   (i) $3 + (-x) = 0$

2. For parts (b) and (h) of problem 1 show that the number you found is the only one making the sentence true.

3. Which of the following sentences are true for all real numbers? (Hint: Remember that the opposite of the sum of two numbers is the sum of their opposites.)

   (a) $-(x + y) = (-x) + (-y)$
   (b) $-x = -(-x)$
   (c) $-(-x) = x$
   (d) $-\left(x + (-2)\right) = (-x) + 2$
   (e) $-\left(a + (-b)\right) = (-a) + b$
   (f) $\left(a + (-b)\right) + (-a) = b$
   (g) $-\left(x + (-x)\right) = x + (-x)$

4. Give a proof (an argument in the manner of the text) that the sentence of 3-(d) is true for all real numbers.

5. In the following proof supply justifications for each step: For all numbers x, y and z, $(-x) + \big(y + (-z)\big) = y + \big(-(x + z)\big)$.

$$\begin{aligned} \text{Proof: } (-x) + \big(y + (-z)\big) &= (-x) + \big((-z) + y\big) \\ &= \big((-x) + (-z)\big) + y \\ &= \big(-(x + z)\big) + y \\ &= y + \big(-(x + z)\big). \end{aligned}$$

6. Using the definition of addition of real numbers, prove the addition property of 0.

6 - 3. Multiplication of real numbers. Now let us decide how we should multiply two real numbers. All that we can say at present is that we know how to multiply two positive numbers or 0.

Of primary importance here is that we maintain the "structure" of the number system. We know that if a, b, c are any numbers of arithmetic, then

$$\begin{aligned} ab &= ba, \\ (ab)c &= a(bc), \\ (a)(1) &= a, \\ (a)(0) &= 0, \\ a(b + c) &= ab + ac. \end{aligned}$$

(What names did we give to these properties of multiplication?) Now whatever meaning we give to the product of two real numbers, we must be sure that these properties of multiplication still hold for all real numbers.

Let us consider some possible products:

$$(2)(3),\quad (3)(-2),\quad (-2)(-3),\quad (-3)(0),\quad (0)(0),\quad (3)(0).$$

(Do these include examples of every case of multiplication of real numbers?) We may immediately dispose of all but the second, third and fourth examples because

$$(2)(3) = 6,\quad (0)(0) = 0,\quad (3)(0) = 0.$$

This leaves the questions:

$$(3)(-2) = ?,\quad (-2)(-3) = ?,\quad (-3)(0) = ?$$

The third question must be answered "$(-3)(0) = 0$" if we want the multiplication property of 0 to be true for all real numbers. The remaining two questions can be answered as follows:

$0 = (3)(0)$

$0 = (3)\big(2 + (-2)\big)$, by writing $0 = 2 + (-2)$

$0 = (3)(2) + (3)(-2)$, if the distributive property

$0 = 6 + (3)(-2)$. must hold for real numbers.

Thus, if we want the properties of multiplication to hold for real numbers, then $(3)(-2)$ must be -6. Why?

Next, we take a similar course to answer the second question.

$0 = (-2)(0)$, if the multiplication property of 0 must hold for real numbers;

$0 = (-2)\big(3 + (-3)\big)$, by writing $0 = 3 + (-3)$;

$0 = (-2)(3) + (-2)(-3)$, if the distributive property must hold for real numbers;

$0 = (3)(-2) + (-2)(-3)$, if the commutative property must hold for real numbers;

$0 = (-6) + (-2)(-3)$, by the previous result,

$$(3)(-2) = -6.$$

Now we have come to a point where $(-2)(-3)$ must be the opposite of -6; hence, if we want the properties of multiplication to hold for real numbers, then $(-2)(-3)$ must be 6.

Recall that the product of two positive numbers is a positive number. Then what are the values of $|3|\ |2|$ and $|-2|\ |-3|$? How do these compare with $(3)(-2)$ and $(-2)(-3)$? Using the same reasoning, what must we mean by $(-3)(4)$? by $(-5)(-3)$? by $(0)(-2)$? Compare $(-3)(4)$ and $-(|-3||4|)$; $(-5)(-3)$ and $|-5|\ |-3|$.

This is the hint we needed. If we want the structure of the number system to be the same for real numbers as it was for the numbers of arithmetic, we must define the product of two real numbers a and b as follows:

> If a and b are both negative or both non-negative, then $ab = |a||b|$.
>
> If one of the numbers a and b is positive or zero and the other is negative, then $ab = -(|a||b|)$.

It is important to recognize that $|a|$ and $|b|$ are numbers of arithmetic for any real numbers a and b. (Why?) Thus, the product $|a||b|$ is a positive number, and we obtain the product ab as either $|a||b|$ or its opposite. It will help you to remember this by completing the sentences; (Supply the words "positive" "negative", or "zero".)

The product of two positive numbers is a ________ number.

The product of two negative numbers is a ________ number.

The product of a negative and a positive number is a ________ number.

The product of a real number and 0 is ________.

The consistency of these results is illustrated in the following sequence of products (fill in the missing products):

$(3)(2) = 6$

$(2)(2) = 4$

$(1)(2) = 2$

$(0)(2) = 0$

$(-1)(2) =$

$(-2)(2) =$

$(3)(-2) = -6$

$(2)(-2) = -4$

$(1)(-2) = -2$

$(0)(-2) = 0$

$(-1)(-2) =$

$(-2)(-2) =$

Exercises 6 - 3.

1. Calculate the following:

(a) $(-\frac{1}{2})(-4)$

(b) $\left((-\frac{1}{2})2\right)(-5)$

(c) $(-\frac{3}{2})\left((-4) + \frac{2}{3}\right)$

(d) $(-\frac{3}{2})(-4) + (-\frac{3}{2})(\frac{2}{3})$

(e) $(-\frac{1}{3})\left((3)(7)\right)$

(f) $(-\frac{1}{3})\left((-3)(-7)\right)$

(g) $(4)\left((-6) + \frac{7}{4}\right)$

(h) $|-3|(-2) + 4$

(i) $|-3||-2| + (-6)$

(j) $(-3)|-2| + 4$

(k) $(-3)(|-2| + 4)$

(ℓ) $(-2)^2(3)$

(m) $\left((-2)(3)\right)^2$

(n) $(|-3|)^2$

(o) $\left((-3) + 2\right)^2$

(p) $(-3)^2 + (2)^2$

2. Find the values of the following for $x = -2$, $y = 3$, $a = -4$, $b = 5$, $c = -6$:

(a) $2x + 7y$

(b) $3(-x) + \big((-4)y + 7(-a)\big)$

(c) $x^2 + (2xa + a^2)$

(d) $(x + a)^2$

(e) $x^2 + \big(3|c| + (-4)|a|\big)$

(f) $|x + 2| + |(-3) + a| + |a + b|$

(g) $(-2)(|8 + c|) + \big((-2)(|8|) + c\big)$

(h) $(xy + ab)c$

3. Which of the following sentences are true?

(a) $2x + 8 = -12$, for $x = -10$.

(b) $2(-y) + 8 = 28$, for $y = -10$.

(c) $-3\big((-2)(-x)\big) + 8 \neq 20$, for $x = 2$.

(d) $-5\big((-b)(-4) + 30\big) < 0$, for $b = 2$.

(e) $2\big((-c)(-3) + (-6)(-2)\big) \leq 0$, for $c = -4$.

(f) $\big((-3)(-d) + (-6)(4)\big) + (-3)(-2) > -4$, for $d = -5$.

(g) $|x + 3| + (-2)\big(|x + (-4)|\big) \geq 1$, for $x = 2$.

(h) $(y+2)^2 + |(-3) + y| \leq 25$, for $y = -6$.

4. Find the truth sets of the following open sentences:

(a) $2x + (-3)(-4) = 8$

(b) $2(-2) + 2y = 3(-2)$

(c) $x + 2 = (3)(-6) + (-4)(-8)$

(d) $x = (-5)(-6) + (-2)(3)$

(e) $-\big(x + 2(3)\big) = (-3)(2) + (3)(-1)$

(f) $|x + 1| = (6)(-2) + (4)(-3)$

(g) $x > (-4)(-2) + (-5)(2)$.

5. Given the set $S = \{1, -2, -3, 4\}$, find the set P of all products of pairs of elements of S.

6. Given the set R of all real numbers, find the set Q of all products of pairs of elements of R. Is the set of real numbers closed under multiplication?

7. Given the set N of all negative real numbers, find the set T of all products of pairs of elements of N. Is the set of negative real numbers closed under multiplication?

8. Given the set $V = \{1, -2, -3, 4\}$, find the set K of all positive numbers obtained as products of pairs of elements of V.

9. Prove that the absolute value of the product ab is the product $|a| \cdot |b|$ of the absolute values.

6 - 4. Properties of multiplication. When we decided how to define the product of two real numbers, we wanted to maintain the structure of the real numbers. This is another way to say that we wanted the familiar properties of multiplication to remain true for all real numbers. Let us state each of these properties and satisfy ourselves that they are true.

Commutative property of multiplication: For any real numbers a and b,

$$ab = ba.$$

If a and b are both negative, then ab = |a||b|. But |a| and |b| are numbers of arithmetic; hence, |a||b| = |b||a|. (Why?) Thus, ab = ba. Illustrate this result with (-3)(-4).

If one of a and b is positive or 0 and the other is negative, then ab = -(|a||b|). But again |a||b| = |b||a|, so that -(|a||b|) = -(|b||a|). (Remember that if numbers are equal, their opposites are equal.) Thus again ab = ba. Illustrate this result with (-3)(4). With (-3)(0).

Here we have given a complete proof of the commutative property for all real numbers. Notice that we based the proof on the precise definition of multiplication.

In general, we have shown that we may change the order of multiplication without changing the product.

Associative property of multiplication. For any real numbers a, b, c,

$$(ab)c = a(bc).$$

It is not too difficult to show that this property holds for one negative, two negatives, or three negatives; but the task is a bit tedious and we shall be content with a few illustrations. (The interested student might want to try to give a general proof of this property.) Verify that each of the following pairs is the same number:

$$\big((3)(2)\big)(-4), \quad (3)\big((2)(-4)\big);$$
$$\big((3)(-2)\big)(-4), \quad (3)\big((-2)(-4)\big);$$
$$\big((-3)(-2)\big)(-4), \quad (-3)\big((-2)(-4)\big).$$

In general, we agree that in multiplying three numbers we may first form the product of any adjacent pair. The effect of these two properties is to allow us to write products of numbers without grouping symbols and to perform the multiplications in any groups and any orders.

Multiplication property of 1. For any real number a,

$$(a)(1) = a.$$

We know that $a(1) = a$ if a is positive or 0. (Why?) What is $a(1)$ if a is negative? (Recall the definition of multiplication.) Finally, what is $-|a|$ if a is a negative? Put these ideas together and you have proved the property of 1.

Notice that

$$\text{if } (-2)(3) = -6, \text{ then } \big((-2)(3)\big)(-4) = (-6)(-4);$$
$$\text{if } (-5)(-3) = 15, \text{ then } \big((-5)(-3)\big)(\tfrac{1}{3}) = (15)(\tfrac{1}{3}).$$

Notice that since "(-2)(3)" and "(-6)" are names for the same number, we may multiply (-4) by the number and obtain " (-2)(3) (-4)" and "(-6)(-4)" as names for the new number.

In general, we have the

Multiplication property of equality. For any real numbers a, b, and c, if $a = b$, then $ac = bc$.

The final and most important property to complete the structure of real numbers is one which ties together the operations of addition and multiplication.

Distributive property. For any real numbers, a, b, and c,

$$a(b + c) = ab + ac.$$

Again we shall consider only a few examples:

$(5)\big(2 + (-3)\big) = ?$ $\qquad (5)(2) + (5)(-3) = ?$

$(5)\big((-2) + (-3)\big) = ?$ $\qquad (5)(-2) + (5)(-3) = ?$

$(-5)\big((-2) + (-3)\big) = ?$ $\qquad (-5)(-2) + (-5)(-3) = ?$

Let us agree that the distributive property holds for all real numbers. Again, this property can be proved by applying the definition of addition and multiplication to all possible cases.

Exercises 6 - 4a.

1. Explain how the associative and commutative properties could be used to perform the following multiplications in the easiest manner.

   (a) $(\frac{1}{3})(\frac{6}{5})(-21)$ $\qquad$ (d) $(-\frac{2}{3})(\frac{4}{5})(\frac{3}{2})(-\frac{5}{4})$

   (b) $(\frac{2}{3})(-\frac{7}{5})(-\frac{3}{4})$ $\qquad$ (e) $(\frac{1}{5})(-19)(-3)(50)$

   (c) $(-5)(17)(-20)(3)$ $\qquad$ (f) $(-7)(-25)(3)(-4)$

2. Use the distributive property, if necessary, to perform the indicated operations with the minimum amount of work:

   (a) $(-\frac{3}{2})\big((-4) + 6\big)$ $\qquad$ (d) $(-6)(\frac{2}{3}) + (-9)(\frac{2}{3})$

   (b) $(-7)(-\frac{4}{3}) + (-7)(\frac{1}{3})$ $\qquad$ (e) $(-7)(-92) + (-7)(-8)$

   (c) $(-\frac{3}{4})\big((-93) + (-7)\big)$ $\qquad$ (f) $(63)(-6) + (-163)(-6)$

3. If x is a real number, then $(x)(0) = 0$. We call this the multiplication property of 0. Apply this property to compute the following:

   (a) $(-4)\left(\left(-\frac{3}{2}\right)(4) + \left(-\frac{6}{5}\right)(-5)\right)$

   (b) $\left((4)(-6) + (-8)(-3)\right)\left(-\frac{47}{13}\right)$

4. A useful property of multiplication is the following: For any real number a,

$$(-1)a = -a.$$

   Prove this property. (Hint: To show that $(-1)a$ is the opposite of a we must show that $a + (-1)a = 0$. To do this, write $a + (-1)a = (1)a + (-1)a$, apply the distributive property, and use the property of 0.)

5. Use the property that you proved in problem 4 to prove:

   (a) For any real numbers a and b, $(-a)(b) = -(ab)$.

   (b) For any real numbers a and b, $(-a)(-b) = ab$.

6. Use the result of problem 4 to verify that the following sentences are true:

   (a) $-\left((-2) + 1\right) = 2 + (-1)$

   (b) $-\left((-4) + (-2)\right) = 4 + 2$

   (c) $-\left((-3)x + 4y\right) = 3x + (-4)y$, for all x and y.

6 - 5. Use of the multiplication properties. Now that we can multiply real numbers and have at our disposal the properties of multiplication or real numbers, we have a strong basis for dealing with a variety of situations in algebra.

Exercises 6 - 5a.

1. Use the distributive property to perform the following multiplications:

   (a) $3(x + 5)$

   (b) $\big(7 + (-k)\big)a$

   (c) $2(a + b + c)$

   (d) $(-9)(a + b)$

   (e) $\big((-p) + q\big)(-3)$

   (f) $(-1)\big(y + (-z) + 5\big)$

   (g) $(13 + x)y$

   (h) $(-8)\big((-4) + (-m)\big)$

   (i) $(-g)\big(r + 1 + (-s) + (-t)\big)$

2. In doing problem 1, you probably used the property: For any real numbers a and b,

   $$(-a)(-b) = ab.$$

   Find the parts of the problem in which you used it.

3. Use the distributive property to write the following expressions as products:

   (a) $5a + 5b$

   (b) $(-9)b + (-9)c$

   (c) $km + kp$

   (d) $3x + 3y + 3z$

   (e) $4p + 8q$

   (f) $(a + b)x + (a + b)y$

   (g) $2ax + 2ay$

   (h) $(-6)a^2 + (-6)b^2$

   (i) $ca + cb + c$

4. Apply the distributive and other properties to the following:

   (a) $3x + 2x = (3 + 2)x = 5x$

   (b) $12t + 7t$

   (c) $9a + (-15)a$

   (d) $(-5)y + 14y$

   (e) $12z + z$
   $\big($Hint: $z = (1)(z)\big)$

(f) $(-3)m + (-8)m$

(g) $\frac{1}{2}a + \frac{3}{2}a$

(h) $(1.6)b + (-2.4)b$

(i) $(-5)x + 2x + 11x$

(j) $3a + 7y$ (Careful!)

(k) $4p + 3p + 9p$

(ℓ) $8r + (-14)r + 6r$

(m) $6a + (-4)a + 5b + 14b$

The phrases you worked with in the above exercises are called expressions. In an expression which has the form of a sum $A + B$, A and B are called terms of the expression; in an expression of the form $A + B + C$, A, B, and C are called terms; etc. The distributive property is very helpful in simplifying an expression. Thus

$$5a + 8a = (5 + 8)a = 13a$$

is a possible and a desirable simplification. However, in

$$5x + 8y$$

no such simplification is possible. Why?

We may sometimes be able to apply the distributive property to some, but not all, terms of an expression. Thus

$$6x + (-9)x + 11x + 5y = \big(6 + (-9) + 11\big)x + 5y = 8x + 5y\ .$$

We shall have frequent occasion to do this kind of simplification. For convenience we shall call it collecting terms or combining terms. We shall usually do the middle step mentally. Thus

$$15w + (-9)w = 6w.$$

Exercises 6 - 5b.

1. Collect terms in the following expressions:

   (a) $3x + 10x$

   (b) $(-9)a + (-4)a$

   (c) $11k + (-2)k$

   (d) $(-27)b + 30b$

   (e) $17n + (-16)n$

   (f) $x + 8x$

   (g) $(-15)a + a$

   (h) $\frac{7}{8}a + \frac{9}{8}a$

   (i) $5p + 4p + 8p$

   (j) $7x + (-10)x + 3x$

   (k) $12a + 5c + (-2)c$

   (ℓ) $6a + 4b + c$

   (m) $9p + 4q + (-3)p + 7q$

2. What other properties of real numbers besides the distributive property did you use in exercise 1(m)?

3. Find the truth set of each of the following sentences: (Collect terms first.)

   (a) $6x + 9x = 30$

   (b) $12y + (-5)y = 35$

   (c) $(-3)a + (-7)a = 40$

   (d) $x + 5x = 3 + 6x$

   (e) $3y + 8y = -99$

   (f) $(-15)z + 12z = 24$

   (g) $14x + (-14)x = 15$

   (h) $(-3)a + 3a = 0$

   (i) $13k + (-14)k + 9k = 0$

   (j) $x + 2x + 3x = 42$

   (k) $a + 2a = 3$

   (ℓ) $x + 9 = 20$

   (m) $3z + 2 = 14$

   (n) $2y = y + 1$

   (o) $12 = 4y + 2y$

   (p) $z + 3z < 0$

   (q) $(-8)y + 9y \geq 5$

   (r) $|x| + 3|x| < 0$

6 - 6. *Solutions of equations.* In the equation

$$3x + 7 = x + 15$$

we can do no collecting of terms on either side of the equation, and yet it is not obvious what the truth set is without some kind of simplification. The chief complication seems to be that the variable appears on both sides of the equation. Things would be simpler if we could form a new sentence with the same truth set in which the variable occurs on only one side. This is possible to do by using the addition property of opposites, $x + (-x) = 0$, and proceeding as follows:

If there is a number x such that

$$3x + 7 = x + 15$$

is a true sentence, then for that value of x,

$$3x + 7 \quad \text{and} \quad x + 15$$

are the same number. If we add $(-x)$ to this number, we get a new number, and $(3x + 7) + (-x)$ and $(x + 15) + (-x)$ are both the same new number. This means that, for the value of x which makes $3x + 7 = x + 15$ true,

$$(3x + 7) + (-x) = (x + 15) + (-x)$$

will also be true. But this equation can be simplified to

$$2x + 7 = 15.$$

While we are about it, we could similarly add (-7) to the number represented by these two sides of the equation and obtain

$$2x = 8.$$

Next we can apply the multiplication property of equality to obtain the sentence

$$x = 4.$$

(By what number did we multiply both sides of the sentence $2x = 8$?) The number which makes this equation true is 4. But any number which makes our original equation true will also make this last one true. Since 4 seems to be the only number which makes our last equation true, we suspect that it makes the original equation true, and testing it soon shows that this is the case. If x is 4, then on the left,

$$3(4) + 7 = 12 + 7 = 19,$$

and on the right,

$$(4) + 15 = 19;$$

so "$3(4) + 7 = (4) + 15$" is true, and $\{4\}$ is the truth set of our equation. It is customary to say that 4 is the solution of the equation.

As another example, find the truth set of the sentence

$$5y + 8 = 2y + (-10).$$

If there is a number y such that $5y + 8 = 2y + (-10)$ is true, then

$$3y + 8 = (-10),$$

and

$$3y = -18,$$

$$y = -6.$$

If $y = -6$, then on the left,

$$5(-6) + 8 = (-30) + 8 = -22,$$

and on the right,

$$2(-6) + (-10) = (-12) + (-10) = -22.$$

So "$5(-6) + 8 = 2(-6) + (-10)$" is a true sentence and the solution is -6.

Exercises 6 - 6.

1. Find the truth set of each of the following equations, using the form shown in the examples above:
   (a) $2a + 5 = 17$
   (b) $4y + 3 = 3y + 5 + y + (-2)$
   (c) $12x + (-6) = 7x + 24$
   (d) $8x + (-3)x + 2 = 7x + 8$ (Collect terms first.)
   (e) $6z + 9 + (-4)z = 18 + 2z$
   (f) $12n + 5n + (-4) = 3n + (-4) + 2n$
   (g) $15 = 6x + (-8) + 17x$
   (h) $5y + 8 = 7y + 3 + (-2y) + 5$
2. Translate the following into open sentences and find their truth sets:
   (a) Mr. Johnson bought 30 ft. of wire and later bought 55 more feet of the same kind of wire. He found that he paid $4.20 more than his neighbor paid for 25 ft. of the same kind of wire. What was the cost per foot of the wire?
   (b) Four times an integer is ten more than twice the successor of that integer. What is the integer?
   (c) In an automobile race, one driver, starting with the first group of cars drove for 5 hours at a certain speed and was then 120 miles from the finish line. Another driver,

who set out with a later heat, had traveled at the same rate as the first driver for 3 hours and was 250 miles from the finish. How fast were these men driving?

(d) The perimeter of a triangle is 44 inches. The second side is three inches more than twice the length of the third side, and the first side is five inches longer than the third side. Find the lengths of the three sides of this triangle.

(e) If an integer and its successor are added, the result is one more than twice that integer. What is the integer?

(f) In a farmer's yard were some pigs and chickens, and no other creatures except the farmer himself. There were, in fact, sixteen more chickens than pigs. Observing this fact, and further observing that there were 74 feet in the yard, not counting his own, the farmer exclaimed happily to himself--for he was a mathematician as well as a farmer, and was given to talking to himself--"Now I can tell how many of each kind of creature there are in my yard." How many were there? (Hint: Pigs have 4 feet, chickens 2 feet.)

(g) At the target shooting booth at a fair, Montmorency was paid 10¢ for each time he hit the target, and was charged 5¢ each time he missed. If he lost 25¢ at the booth and made ten more misses than hits, how many hits did he make?

(h) The sum of two consecutive odd integers is 11. What are the integers?

6 - 7. Further use of the multiplication properties. We have seen how the distributive property allows us to collect terms of an expression. The properties of multiplication are helpful also in certain techniques of algebra related to products involving algebraic expressions.

Example 1. "(3xy)(7ax)" can be more simply written as "$21ax^2y$."

Give the reasons for each of the following steps which show this is true. (For ease in writing, we shall often use "$\cdot$" instead of "x" to indicate multiplication.)

$$\begin{aligned}(3xy)(7ax) &= 3\cdot x\cdot y\cdot 7\cdot a\cdot x\\ &= 3\cdot 7\cdot a\cdot x\cdot x\cdot y\\ &= (3\cdot 7)a(x\cdot x)y\\ &= 21ax^2y.\end{aligned}$$

While in practice we do not write down all these steps, we must continue to be aware of how this simplification depends on our basic properties of multiplication, and we should be prepared to explain the intervening steps at any time.

Exercises 6 - 7a.

Simplify the following expressions and, in problem 1, write the steps which explain the simplification.

1. $(4y^2)(-3ay)$
2. $(\frac{3}{4}\ abc)(\frac{1}{2}\ bcd)$
3. $(-12pq)(-4pq)$
4. $(20\ b^2c^2)(10\ bd)$

We can combine the method of the preceding exercises with the distributive property to perform multiplications such as the following:

$$(-3a)\big(2a + 3b + (-5\ c)\big) = (-6a^2) + (-9ab) + 15ac\ .$$

Furthermore, since we have shown in section 6-4 that

$$-a = (-1)(a),$$

we may again with the help of the distributive property simplify expressions such as

$$\begin{aligned} -\big(x^2 + (-7x) + (-6)\big) &= (-1)\big(x^2 + (-7x) + (-6)\big) \\ &= (-x^2) + 7x + 6\ . \end{aligned}$$

Exercises 6 - 7b.

Simplify as indicated above:

1. $5x(x + 6)$
2. $10b\big(2b^2 + 7b + (-4)\big)$
3. $-(p + q + r)$
4. $(-7)\big(3a + (-5b)\big)$
5. $6xy(2x + 3xy + 4y)$
6. $-(a^2 + 2ab + b^2)$
7. $(-4c)\big(2a + (-5b) + (-c)\big)$
8. $(-x)\big(x + (-1)\big)$

Carry out the following series of multiplications by the distributive property.

$$a(x + 2) =$$

$$w(x + 2) =$$

$$B(x + 2) =$$

$$\alpha(x + 2) =$$

$$\beta(x + 2) =$$

$$(x + 3)(x + 2) =$$

Do you get for the last one of these the following?

$$\begin{aligned}(x + 3)(x + 2) &= (x + 3)x + (x + 3)2\\ &= x^2 + 3x + 2x + 6\\ &= x^2 + 5x + 6.\end{aligned}$$

By such successive uses of the distributive property we can perform multiplications involving several terms.

Example 2. $$\begin{aligned}\big(a + (-7)\big)(a + 3) &= \big(a + (-7)\big)a + \big(a + (-7)\big)3\\ &= a^2 + (-7)a + 3a + (-21)\\ &= a^2 + (-4)a + (-21).\end{aligned}$$

Example 3. $$\begin{aligned}(x + y + z)(b + 5) &= (x + y + z)b + (x + y + z)5\\ &= bx + by + bz + 5x + 5y + 5z.\end{aligned}$$

Exercises 6 - 7c.

1. In Example 2 the distributive property was used four times. Point out all four places where it is used.

2. Perform the following multiplications.

(a) $(x + 8)(x + 2)$

(b) $\big(y + (-3)\big)\big(y + (-5)\big)$

(c) $\big(6a + (-5)\big)\big(a + (-2)\big)$

(d) $(a + 1)(a^2 + 3a + 4)$

(e) $(x + 6)\big(z + (-6)\big)$

3. Show that for real numbers a, b, c, d,

$$(a + b)(c + d) = ac + (bc + ad) + bd.$$

(Notice that ac is the product of the first terms, bd is the product of the second terms, and (bc + ad) is the sum of the mixed products.

4. Use the formula developed in problem 3, where possible, to multiply the following:

   (a) $(a + 3)(a + 1)$

   (b) $(2x + 3)(3x + 4)$

   (c) $(a + c)(b + d)$

   (d) $\left(y + (-4)\right)\left(y^2 + (-2y) + 1\right)$

   (e) $(m + 3)(m + 3)$

6 - 8. Order properties. Compare the numbers $(-\frac{35}{12})$ and $(-\frac{35}{12}) + 2$. On the number line we see that

$$(-\frac{35}{12}) < (-\frac{35}{12}) + 2:$$

Now compare $\frac{43}{13}$ and $\frac{43}{13} + 2$. In general, compare z and z + 2, where z is any real number. We know that exactly one of three possibilities holds:

$$z < z + 2, \quad z = z + 2, \quad \text{or } z + 2 < z. \quad \text{(Why?)}$$

Which of these three is true for any z? Why? Does your answer depend on the value of z?

Compare in the same way the two numbers $\frac{31}{8}$ and $\frac{31}{8} + (-2)$; $-\frac{4}{7}$ and $(-\frac{4}{7}) + (-2)$; z and $z + (-2)$.

By this time you should conclude that adding a positive number to any real number z gives a sum greater than z, whereas adding a negative number to any real number z gives a sum less than z. Is every number greater than z obtained as the sum of z and a positive number? Consider the number $z = -5$ and the greater number $(-\frac{3}{2})$. Is there a positive number b such that $(-5) + b = -\frac{3}{2}$? What is it?

Let us generalize these ideas. If a and b are real numbers and b is positive, then $a + b$ is obtained on the number line by moving to the right from a. Since "is to the right of" means "is greater than", we have:

(1) $a + b > a$, if b is positive.

On the other hand, if we know that $a + b > a$, then $a + b$ is obtained by moving to the right from a on the number line. This can happen only if b is positive. In other words:

(2) $a + b > a$, only if b is positive.

Combining (1) and (2), we obtain

(3) $a + b > a$, if and only if b is positive.

Thus, sentence (3) is a convenient way of writing sentences (1) and (2).

Now we are ready to state a general property of order of numbers:

> If a and c are any two real numbers such that $c > a$, then there is a positive real number b such that
>
> $$c = a + b.$$

By this time you are probably convinced that this general property is true. What is the positive number b such that $-\frac{5}{3} = (-6) + b$? Such that $\frac{4}{3} = (-\frac{4}{3}) + b$? Use the number line to help you if you need to.

However, assuming that the property is true, how would you go about finding the number b in general? Since $c = a + b$, add $(-a)$ to both sides by the addition property of equality, etc. You finish it.

Now we have a new view of the relation "$>$". Any sentence involving "$>$", such as

$$-3 > -5$$

can now be replaced by a sentence involving "=", in this case

$$-3 = (-5) + 2,$$

where a positive number is added to the smaller to obtain the greater. Is the converse true; that is, if b is positive and $c = a + b$, is it true that $c > a$? Why? This new ability to "translate" back and forth between a sentence involving "$>$" and a sentence involving "=" will be useful in proving results about inequalities.

Exercises 6 - 8a.

1. For each pair of numbers, determine their order and find the positive number b which when added to the smaller gives the larger

(a) -15 and -24

(b) $\frac{63}{4}$ and $-\frac{5}{4}$

(c) $\frac{6}{5}$ and $\frac{7}{10}$

(d) $-\frac{1}{2}$ and $\frac{1}{3}$

(e) -254 and -345

(f) $-\frac{33}{13}$ and $-\frac{98}{39}$

(g) 1.47 and -0.21

(h) $(-\frac{2}{3})(\frac{4}{5})$ and $(\frac{3}{2})(-\frac{5}{4})$

2. Show that the following is a true statement:

If a and b are real numbers, then

$$a + b < a, \text{ if and only if } b \text{ is negative.}$$

(Hint: Follow the similar discussion for b positive.)

3. State a general property of order which gives the meaning of "$a < c$" in terms of an equation.

4. Which of the following sentences are true for all values of the variables?

(a) If $a + 1 = b$, then $b > a$.

(b) If $a + (-1) = b$, then $a < b$.

(c) If $(a + c) + 2 = (b + c)$, then $a + c < b + c$.

(d) If $(a + c) + (-2) = (b + c)$, then $a + c > b + c$.

(e) If $a < -2$, then there is a positive number d such that $-2 = a + d$.

(f) If $a > -2$, then there is a positive number d such that $a = (-2) + d$.

5. (a) If $5 + 8 = 13$, write two true sentences involving "$<$" relating the numbers 5, 8, 13.

(b) If $(-3) + 2 = (-1)$, how many true sentences involving "$<$" can you write.

(c) If $5 < 7$, write two true sentences involving "=" relating the numbers 5, 7.

6. Show on the number line that if a and c are real numbers and if b is a negative number such that $c = a + b$, then $c < a$.

Let us assume that

$$\frac{220}{363} < \frac{112}{184}$$

is a true sentence. (Don't bother checking; take our word for it.) What can you conclude about the order of each of the following pairs of numbers?

$$\frac{220}{363} + (-3) \text{ and } \frac{112}{184} + (-3),$$

$$\frac{220}{363} + \frac{17}{20} \text{ and } \frac{112}{184} + \frac{17}{20},$$

$$\frac{220}{363} + (-\sqrt{2}) \text{ and } \frac{112}{184} + (-\sqrt{2}),$$

$$\frac{220}{363} + 0 \text{ and } \frac{112}{184} + 0.$$

We see here a property of inequalities which seems to be true for all real numbers:

Addition property of order. If a, b and c are real numbers and if $a < b$, then

$$a + c < b + c.$$

State in your own words why you think this general property is true. What does it mean in terms of points on the real number line? What does it mean in terms of the distance from a to b and from $a + c$ to $b + c$ on the number line?

The following is a proof of the order property. Can you fill in the reasons for each step?

If $a < b$, then there is a positive number d such that

$$b = a + d;$$

then $b + c = (a + d) + c$, (Why?)

$(b + c) = (a + c) + d$. (Why?)

But d is positive; hence

$a + c < b + c$. (Why?)

Exercises 6 - 8b.

1. By applying the order property of addition, determine which of the following sentences are true:

   (a) $(-\frac{6}{5}) + 4 < (-\frac{3}{4}) + 4$

   (b) $(-\frac{5}{3})(\frac{6}{5}) + (-5) > (-\frac{5}{2}) + (-5)$

   (c) $(-5.3) + (-2)(-\frac{4}{3}) < (-0.4) + \frac{8}{3}$

   (d) $(\frac{5}{2})(-\frac{3}{4}) + 2 \geq (-\frac{15}{8}) + 2$

2. Formulate an order property of addition for each of the relations "$\leq$", "$>$", "$\geq$".

3. An extension of the order property states that:

   If a, b, c, d are real numbers such that

   $a < b$ and $c < d$, then $a + c < b + d$.

   This can be proved in three steps. Fill in the reason for each step:

If $a < b$, then $a + c < b + c$;
if $c < d$, then $b + c < b + d$;
hence, $a + c < b + d$.

4. Find the truth set of each of the following sentences:

Example: $(-\frac{3}{2}) + x < (-5) + \frac{3}{2}$.

$x < (-5) + \frac{3}{2} + \frac{3}{2}$, add $\frac{3}{2}$ to both sides,

$x < -2$, add terms on right.

Thus, if x is a number which makes the original sentence true, then $x < -2$. If $x < -2$, is the original sentence true? How can you show this?

(a) $\frac{3}{5} + (-\frac{3}{10}) < x + (-\frac{4}{5})$

(b) $3x > \frac{4}{3} + 2x$

(c) $(-\frac{2}{3}) + 2x \geq \frac{5}{3} + x$

(d) $(-x) + 4 < (-3) + |-3|$

(e) $(-5) + (-x) < \frac{2}{3} + |-\frac{4}{3}|$

(f) $(-2) + 2x < (-3) + 3x + 5$

(g) $(-\frac{3}{4}) + \frac{5}{4} \geq 2x + |-\frac{3}{2}|$.

5. Graph the truth sets of parts (a), (e), and (g) of problem 4.

6. Which of the following sentences are true for all values of the variables?

(a) If $b < 0$, then $3 + b < b$.

(b) If $b < 0$, then $3 + b < 3$.

(c) If $x < 2$, then $2x < 4$.

7. Verify that each of the following is true:

   (a) $|3 + 4| \leq |3| + |4|$.

   (b) $|(-3) + 4| \leq |-3| + |4|$.

   (c) $|(-3) + (-4)| \leq |-3| + |-4|$

   (d) State a general property relating $|a + b|$, $|a|$ and $|b|$ for any real numbers a and b.

8. What general property can be stated for multiplication similar to the property for addition in problem 7?

9. Trarslate the following into open sentences and find their truth sets:

   (a) When Joe and Moe were planning to buy a sailboat, they asked a salesman about the cost of a new type of a boat that was being designed. The salesman replied, "It won't cost more than $380". If Joe and Moe had agreed that Joe was to contribut $130 more than Moe when the boat was purchased, how much would Moe have to pay?

   (b) Three more than six times a number is greater than seven increased by four times the number. What is the number?

   (c) A teacher says, "If I had three times as many students in my class as I do have, I would have a least 26 more than I now have". How many students does he have in his class?

   (d) A student has test grades of 82 and 91. What must he score on a third test to have an average of 90 or higher?

(e) Bill is 5 years older than Norma, and the sum of their ages is less than 23. How old is Norma?

10. Prove: For real numbers a and b

$$a + b < a \text{ if and only if } b \text{ is negative.}$$

(Hint: If b is negative, then $b < 0$. Now use the addition property of order.)

Now that we have determined the order of $a + c$ and $b + c$ when $a < b$, let us ask about the order of ac and bc when $a < b$. Notice that:

If $-3 < -2$, then $(-3)(2) < (-2)(2)$.

If $-2 < 4$, then $(-2)(3) < (4)(3)$.

If $3 < 5$, then $(3)(6) < (5)(6)$.

If $3 < 5$, then $3(-2) > 5(-2)$.

In these cases, if $a < b$, then $ac < bc$, provided c is a positive number. This is a very important property of order. How would you use it to tell quickly whether the following sentences are true?

If $\frac{1}{4} < \frac{2}{7}$, then $\frac{5}{4} < \frac{10}{7}$.

If $-\frac{5}{6} < -\frac{14}{17}$, then $-\frac{5}{18} < -\frac{14}{51}$.

What positive number did you multiply both members by in each of these cases?

But this property tells us nothing about the order of ac and bc if c is a negative number. Do you think there is a corresponding property? Let us try to find a property this time without looking at special examples.

We know a property when c is positive. But we are faced with negative c. Well, if $c < 0$, then by taking opposites, $-c > 0$. This is just what we want, because now $(-c)$ is positive, and our property says that if $a < b$, then

$$a(-c) < b(-c).$$

This we can change to

$$-(ac) < -(bc).$$

But then, by again taking opposites, we have

$$ac > bc.$$

Thus we have discovered another property of order: If $a < b$ and c is negative, then $ac > bc$.

These are usually stated together as the:

> Multiplication property of order. If a, b and c are real numbers and if $a < b$, then
>
> $ac < bc$, if c is positive,
>
> $ac > bc$, if c is negative.

Don't forget the clue: If we multiply both sides of an inequality by a positive number, the order remains unchanged. But if we multiply both sides by a negative number, the order is reversed. What happens if we multiply by 0?

Exercises 6 - 8c.

1. Find the truth set of each of the following sentences:

   Example: $(-3)x + 4 < -5$.

   $(-3)x < (-5) + (-4)$, add (-4) to both sides,

$(-3)x < -9$, add terms,

$(-\frac{1}{3})(-3)x > (-\frac{1}{3})(-9)$, multiply both sides by $(-\frac{1}{3})$ (note the reversed order),

$x > 3$, multiply numbers.

Thus, if x is a number which makes the original sentence true, then $x > 3$. If $x > 3$, show that the original sentence is true.

(a) $(-4) + 7 < (-2)x + (-5)$

(b) $4x + (-3) > 5 + (-2)x$

(c) $(\frac{2}{3}) + (-\frac{5}{6}) < (-\frac{1}{6}) + (-3)x$

(d) $\frac{1}{2}x + (-2) < (-5) + \frac{5}{2}x$

(e) $2x < 3 + (-2)(-\frac{4}{3})$

(f) $4x + 7 + (-2)x > (-2) + 5 + (-3)x$

(g) $-(2 + x) < 3 + (-7)$

2. Graph the truth sets of parts (a) and (b) of problem 1.

3. Translate the following into open sentences and solve:

   (a) If a rectangle has area 12 square inches and one side has length less than 6 inches, what is the length of the adjacent side?

   (b) If a rectangle has area 12 square inches and one side has length between 4 and 6 inches, what is the length of the adjacent side?

4. If $x \neq 0$, then x is either negative or positive. If x is positive, then what kind of number is x^2? If x is negative,

what about x^2? State a general result about x^2 if $x \neq 0$. What is a general result about x^2 for any real number x?

6 - 9 Summary. Let us list the properties of real numbers that we have discovered.

If a and b are real numbers, then what do we mean by $a + b$? With this meaning of addition, the following properties of addition hold:

For real numbers a, b and c,

(1) $a + b = b + a$,

(2) $(a + b) + c = a + (b + c)$,

(3) $a + 0 = a$,

(4) $a + (-a) = 0$,

(5) $a + b = 0$ if and only if $b = -a$,

(6) $-(a + b) = (-a) + (-b)$,

(7) $-(-a) = a$,

(8) if $a = b$, then $a + c = b + c$,

(9) $c > a$ if and only if there is a positive number b such that $c = a + b$,

(10) if $a < b$ then $a + c < b + c$,

Some of these properties of addition have been given names. State each one in words and refer to it by its name.

If a and b are real numbers, then what do we mean by ab? With this meaning of multiplication, the following properties of multiplication hold:

For real numbers a, b, and c,

(1) $ab = ba$,

(2) $(ab)c = a(bc)$,

(3) $a(1) = a$,

(4) $a(0) = 0$,

(5) $a(-b) = -(ab)$,

(6) $(-a)(-b) = ab$,

(7) $(-1)a = -a$,

(8) if $a = b$, then $ac = bc$,

(9) if $a < b$ and $c > 0$, then $ac < bc$,

(10) if $a < b$ and $c < 0$, then $ac > bc$,

(11) if $a \neq 0$, then $a^2 > 0$.

What names did we give to these properties? State the properties in words.

The most important property of all, the one relating addition and multiplication is

(12) $a(b + c) = ab + ac$.

Chapter 7

Subtraction and Division of Real Numbers

7 - 1. **Meaning of subtraction.** In arithmetic we did a great deal of subtracting. Perhaps you called it "taking away". Now that we are becoming familiar with real numbers, it will be convenient to have an operation of subtraction for these numbers, also.

Exercises 7 - 1a. Compare the results of the two operations:

1. Add the opposite of 9 to 20. Subtract 9 from 20.
2. Add the opposite of $\frac{3}{4}$ to $\frac{7}{8}$. Subtract $\frac{3}{4}$ from $\frac{7}{8}$.
3. Add the opposite of 0.76 to 1.82. Subtract 0.76 from 1.82.
4. Add -50 to 400. Subtract 50 from 400.

After thinking about the above exercises, you are probably ready to define subtraction for the real numbers.

> To subtract the real number b from the real number a, add the opposite of b to a. The result of subtractin b from a is denoted by a - b.

Thus, for real numbers a and b,

$$a - b = a + (-b).$$

If we apply this definition to 20 - 9, we have

$$20 - 9 = 20 + (-9) = 11.$$

Does this agree with ordinary subtraction in arithmetic?

What would we have said in arithmetic if someone had asked us to find 9 - 20? Now we can say

$$9 - 20 = 9 + (-20) = -11.$$

Thus we can do subtractions which were not possible in arithmetic. Furthermore, we can subtract when one or both of the numbers are negative.

What is the opposite of 9? The opposite of -9? The opposite of -20? Notice how these opposites are used in the following examples.

$$20 - (-9) = 20 + 9 = 29$$
$$(-20) - 9 = (-20) + (-9) = -29$$
$$(-20) - (-9) = (-20) + 9 = -11.$$
$$9 - (-20) = 9 + 20 = 29$$

You probably noticed in the above examples that the symbol "-" is used in two ways. In "20 - (-9)", what is the meaning of the first "-"? What is the meaning of the second "-"? To help keep these uses of the symbol clear, we make the following parallel statements about them.

In "a - b",	In "a + (-b)"
"-" stands between two numerals and indicates the operation of subtraction.	"-" is part of one numeral and indicates the opposite of.

Exercises 7 - 1b.

1. Subtract -8 from 15.
2. From -25, subtract -4.
3. What number is 6 less than -9?
4. From 22, take away -30.
5. -12 is how much greater than -17?
6. How much greater is 8 than -5?
7. $(-5000) - (-2000)$
8. $\frac{3}{4} - (-\frac{1}{2})$
9. $(-\frac{9}{2}) - (-6)$
10. $(-0.631) - (0.631)$
11. $(-1.79) - 1.22$
12. $0 - (-5)$
13. $75 - (-85)$
14. $(-\frac{5}{9}) - \frac{13}{9}$
15. Let R be the set of all real numbers, and S the set of all numbers obtained by performing the operation of subtraction on pairs of numbers of R. Is S a subset of R? Are the real numbers closed under subtraction? Were the numbers of arithmetic closed under subtraction?
16. Is $a - b = b - a$ a true statement for all real numbers? Can you find any real numbers for which it is true? Is subtraction commutative? Is subtraction associative?
17. Show why $a - a = 0$ is true for all real numbers.
18. Find the truth set of each of the following equations:
    - (a) $y - 725 = 25$
    - (b) $z - 34 = 76$
    - (c) $2x + 8 = -16$.
    - (d) $3y - 2 = -14$
    - (e) $x + 23.6 = 7.2$
    - (f) $z + (-\frac{3}{4}) = -\frac{1}{2}$
19. From a temperature of 3° below zero, the temperature dropped 10°. What was the new temperature? Show how this question is related to subtraction of real numbers.

20. Mrs. J. had a credit of $7.23 in her account at a department store. She bought a dress for $15.50 and charged it. What was the balance in her account?

21. Billy owed his brother 80 cents. He repaid 50 cents of the debt. How can this transaction be written as a subtraction of real numbers?

22. The bottom of Death Valley is 282 feet below sea level. The top of Mt. Whitney, which can be seen from Death Valley, has an altitude of 14,495 feet. How high above Death Valley is Mt. Whitney?

23. What number added to -4.6 will give -7.8 as a result?

24. How far is it on the number line from $3\frac{1}{2}$ to $-2\frac{1}{2}$?

25. If subtracting 10 from 50 on the number line is done by moving 10 units to the left from 50, how would we subtract (-10) from 50 on the number line?

26. We think of addition of real numbers on the real number line as moving to the right when adding a positive number, to the left when adding a negative number. What would be the procedure on the number line to subtract a positive number from another real number? Illustrate by performing the following subtractions on the number line.

(a) 9 - (-2)
(b) (-3) - 4
(c) (-3) - (-8)
(d) 10 - 15
(e) 7 - (-8)
(f) (-5) - 2
(g) (-8) - (-1)
(h) 0 - (-7)

27. Consider the statement: For all real numbers a, b, and c, $a - b = c$ if and only if $a = b + c$. State in words the two separate statements included here, one involving "if" and the other "only if". How can these statements be proved by using the addition property of equality?

7 - 2. Absolute values and subtraction. We will find many uses for the absolute value of the difference of two numbers as illustrated in the following exercises.

Exercises 7 - 2.

1. Perform the indicated operations:

   (a) $|5| - |3|$

   (b) $|3| - |5|$

   (c) $|-5| - |-3|$

   (d) $|5 - 3|$

   (e) $|3 - 5|$

   (f) $|5 - 5|$

   (g) $|5| + |-5|$

   (h) $|5| - |5|$

   (i) $|-5| - |5|$

   (j) $|(-5) - 3|$

   (k) $|-3| + |5|$

   (l) $|3| - |-3|$

2. On the real number line, locate 2, 8, and $(2 - 8)$. What must be added to 8 to yield 2? How can you indicate this in terms of distance from one point to another on the number line?

3. On the real number line, given the points a, b, how can you describe $a - b$ in terms of a distance from one of these points to the other? Illustrate with $(-3) - (-2)$. With $(-3) - 2$. With $(-3) - (-6)$.

4. How can we interpret $|8 - 2|$ on the number line? What about $|2 - 8|$? What does $(2 - 8)$ tell about a distance on the number line that $|2 - 8|$ does not tell? If we were interested in expressing the distance between a and b on the number

line and did not care about direction on the line, would $|b - a|$ serve our purpose? How can we express the distance from a to b as a difference?

5. For each of the following pairs of expressions, fill in the symbol ">", "=", or "<", which will make a true sentence.

(a) $|9 - 2| \ ? \ |9| - |2|$

(b) $|2 - 9| \ ? \ |2| - |9|$

(c) $|9 - (-2)| \ ? \ |9| - |-2|$

(d) $|(-2) - 9| \ ? \ |-2| - |9|$

(e) $|(-9) - 2| \ ? \ |-9| - |2|$

(f) $|2 - (-9)| \ ? \ |2| - |-9|$

(g) $|(-9) - (-2)| \ ? \ |-9| - |-2|$

(h) $|(-2) - (-9)| \ ? \ |-2| - |-9|$

6. Write a symbol between $|a - b|$ and $|a| - |b|$ which will make a true sentence for all real numbers a and b. Do the same for $|a - b|$ and $|b| - |a|$. For $|a - b|$ and $\big||a| - |b|\big|$.

7. Describe the resulting sentences in problem 6 in terms of distances on the number line.

8. What are the two numbers x on the number line such that

$$|x - 4| = 1,$$

that is, the two numbers x such that the distance between x and 4 is 1?

9. What is the truth set of the sentence

$$|x - 4| < 1,$$

that is, the set of numbers x such that the distance between x and 4 is less than 1? Graph this set on the number line.

10. Graph the truth set of

$$x > 3 \quad \text{and} \quad x < 5$$

on the number line. Is this set the same as the truth set of $|x - 4| < 1$? (We usually write "$3 < x < 5$" for the sentence "$x > 3$ and $x < 5$.")

11. Find the truth set of each of the following equations; graph each of these sets:

    (a) $|x - 6| = 8$

    (b) $y + |-6| = 10$

    (c) $|10 - a| = 2$

    (d) $|x| > 3$

    (e) $|v| > -3$

    (f) $|y| + 12 = 13$

    (g) $|y - 8| < 4$ (Read this: The distance between y and 8 is less than 4.)

    (h) $|z| + 12 = 6$

    (i) $|x - (-19)| = 3$

    (j) $|y + 5| = 9$

12. For each sentence in the left column pick the sentence in the right column which has the same truth set:

    $|x| = 3$
    $|x| < 3$
    $|x| \leq 3$
    $|x| > 3$
    $|x| \geq 3$
    $|x| \not< 3$
    $|x| \nleq 3$
    $|x| \not> 3$
    $|x| \ngeq 3$

    $x = -3$ and $x = 3$
    $x = -3$ or $x = 3$
    $x > -3$ and $x < 3$
    $x > -3$ or $x < 3$
    $x < -3$ and $x > 3$
    $x < -3$ or $x > 3$
    $x \geq -3$ and $x \leq 3$
    $x \leq -3$ or $x \geq 3$

13. Perform the indicated operation, using the distributive law where necessary.

(a) $a^2 + 3a^2$

(b) $\pi - (-\pi)$

(c) $8k - (-11k)$

(d) $6\sqrt{2} - 9\sqrt{2}$

(e) $6x - 2x$

(f) $9x^2 + (-4x^2)$

(g) $0 - 5a$

(h) $(-25pq) - pq$

(i) $(-12y) - 4y$

(j) $(-3b) - (-3b)$

(k) $(-4y) - 0$

(l) $0 - (-3m)$

14. The temperature drops 15° from an initial temperature of 4° above zero. Express this statement as a subtraction of real numbers.

15. A submarine has been cruising at 50 feet below the surface. It then goes 30 feet deeper. Express this change as a subtraction of real numbers.

16. -16 is 25 less than some number. Find the number.

17. If the time at 12 o'clock midnight is considered as the starting time, that is, at 12 o'clock midnight $t = 0$, what is the time interval from 11 o'clock P.M. to 2 o'clock A.M.? From 6 o'clock A.M. to 4 o'clock A.M. the next day?

18. From a point marked 0 on a straight road, John and Rudy ride bicycles. John rides 10 miles per hour and Rudy rides 12 miles per hour. Find the distance between them after

(1) 3 hours, (2) $1\frac{1}{2}$ hours, (3) 20 minutes, if

(a) They start from the 0 mark at the same time and John goes east and Rudy goes west.

(b) John is 5 miles east and Rudy is 6 miles west of the 0 mark when they start and they both go east.

(c) John starts from the 0 mark and goes east. Rudy starts from the 0 mark 15 minutes later and goes west.

(d) Both start at the same time. John starts from the 0 mark and goes west and Rudy starts 6 miles west of the

0 mark and also goes west.

7 - 3. Subtraction of a sum. We shall sometimes need to do subtractions such as

$$(7x^2 - 3x) - (4x^2 - 7x - 8).$$

Subtracting $(4x^2 - 7x - 8)$ is the same as adding $-(4x^2 - 7x - 8)$. Why? We proved, in Chapter 6, that $-(a + b) = (-a) + (-b)$ for all real numbers a and b. By an extension of this property, we may write

$$\begin{aligned} -(4x^2 - 7x - 8) &= (-4x^2) + \big(-(-7x)\big) + \big(-(-8)\big) \\ &= -4x^2 + 7x = 8. \qquad \text{Why?} \end{aligned}$$

Therefore,

$$(7x^2 - 3x) - (4x^2 - 7x - 8) = (7x^2 - 3x) + (-4x^2 + 7x + 8)$$

All we need to do now is to rearrange and regroup these terms by the commutative and associative properties of addition to obtain

$$\begin{aligned} (7x^2 - 3x) - (4x^2 - 7x - 8) &= \big(7x^2 + (-4x^2)\big) + \big((-3x) + 7x\big) + 8 \\ &= 3x^2 + 4x + 8 \end{aligned}$$

In actual practice, we write all this more compactly:

$$(7x^2 - 3x) - (4x^2 - 7x - 8) = 7x^2 - 3x - 4x^2 + 7x + 8$$

You may be impressed by the way we are now doing a number of steps mentally. This ability to comprehend several steps without writing them all down is a sign of our mathematical growth. We must be careful, however, to be able, at any time, to pick out all the detailed steps and explain each one.

For instance, give the reason for each of the following steps:

$$
\begin{aligned}
(6a - 8b + c) - (4a - 2b + 7c) \\
&= \big(6a + (-3b) + c\big) + -(4a + (-2b) + 7c) \\
&= \big(6a + (-8b) + c\big) + (-4a) + (-(-2b)) + (-7c) \\
&= \big(6a + (-8b) + c\big) + -4a + 2b + (-7c) \\
&= \big(6a + (-4a)\big) + (-8b) + 2b + c + (-7c) \\
&= \big(6 + (-4)\big)a + \big((-8) + 2\big)b + \big(1 + (-7)\big)c \\
&= 2a + (-6)b + (-6)c \\
&= 2a + (-6b) + (-6c) \\
&= 2a - 6b - 6c
\end{aligned}
$$

Exercises 7 - 3a.

Perform the following operations:

1. $(7y^2 + 2y - 9) - (3y^2 - 8y + 4)$
2. $(5a - 17b) - (-4a - 6b)$
3. $(x^2 + 2xy + y^2) - (x^2 - 2xy + y^2)$
4. $(2x + 7) + (4x^2 + 8 - x)$
5. $(3\pi + 9) - (5\pi - 9)$
6. $(2\sqrt{5} - 8) - (\sqrt{5} - 2)$
7. $(6 + 2\sqrt{3}) - (-8 + 4\sqrt{3})$
8. $(6x^3 + 5x^2 - 6) - (2x^2 - 3x + 9)$
9. $(3a + 2b - 4) - (5a - 3b + c)$
10. $(4x - 7 - 8x^2) - (8 + 3x^2 + 5x)$
11. $(7xy - 4xz) - (8xy - 3yz)$
12. $(-4 + 7c - 2) - (5a + 3c + 7)$

13. We proved in Chapter 6 that $-a = (-1)a$ for all real numbers a. Thus, $-(4x^2 - 7x - 8) = (-1)(4x^2 - 7x - 8)$, and this suggests the distributive property. Justify each step in the following:

$$\begin{aligned}
(7x^2 - 3x) - (4x^2 - 7x - 8) \\
&= (7x^2 - 3x) + \left(-(4x^2 - 7x - 8)\right) \\
&= (7x^2 - 3x) + (-1)(4x^2 - 7x - 8) \\
&= (7x^2 - 3x) + (-1)\left(4x^2 + (-7x) + (-8)\right) \\
&= (7x^2 - 3x) + \left((-1)(4x^2) + (-1)(-7x) + (-1)(-8)\right) \\
&= (7x^2 - 3x) + (-4x^2 + 7x + 8).
\end{aligned}$$

As before, we write all this more compactly as

$$(7x^2 - 3x) - (4x^2 - 7x - 8) = 7x^2 - 3x - 4x^2 + 7x + 8,$$

performing all the intermediate steps mentally.

14. Use the distributive property, as in problem 11, to perform the following subtractions, showing the steps in detail in part (a):

(a) $(5a - 17b) - (-4a - 6b)$

(b) $(6x^3 + 5x^2 - 6) - (2x^2 - 3x + 9)$

There are times when it is convenient to arrange subtractions, such as those in the preceding paragraph, as follows:

$$\begin{array}{r}
9a^2 - 5a + 6 \\
\underline{2a^2 - 8a - 7} \\
7a^2 + 3a + 13
\end{array}$$

The reasoning is, of course, the same as before.

Exercises 7 - 3b.

Perform the subtractions in problems 1 to 6, arranging your work with terms beneath each other as shown in the example above. Note that we put beneath each other terms to which the distributive law can be applied.

1. $(19x^2 + 12x - 15) - (20x^2 - 3x - 1)$
2. $(8a - 13) - (7a + 12)$
3. $(14a^2 - 5a + 1) - (6a^2 - 9)$
4. $(3n + 12p - 8a) - (5a - 7n - p)$
5. $(7x^2 - 7) - (3x + 9)$
6. $(a^2 - b^2) - (a^2 - 2ab + b^2)$
7. From $11a + 13b - 7c$ subtract $8a - 5b - 4c$.
8. What is the result of subtracting $-3x^2 + 5x - 7$ from $-3x + 12$?
9. What must be added to $3s - 4t + 7u$ to obtain $-9s - 3u$?
10. If $a > b$, what can you say of $a - b$? Prove your statement. (Read this: If a is to the right of b, then the difference from b to a is __________.)
11. If $(a - b)$ is a positive number, which of the statements, $a < b$, $a = b$, $a > b$, is true? What if $(a - b)$ is a negative number? What if $(a - b)$ is zero?
12. If a, b, and c are real numbers and $a > b$, what can we say about the order of $a - c$ and $b - c$? Prove your statement.

7 - 4. Multiplicative inverse. Since $6 + (-6) = 0$, we called -6 the additive inverse of 6. That is, -6 is the number which when added to 6 yields the sum 0. Is there a

similar relation among numbers with respect to multiplication? We know that the product of 0 and any number is 0. But what is a number which when multiplied by 6 yields the product 1?

By experiment or by former knowledge of arithemtic, you will probably say, "$\frac{1}{6}$ is such a number, because $6 \times \frac{1}{6} = 1$." What is a number which when multiplied by -2 yields the product 1? Answer the same question for $-\frac{1}{3}$. For $\frac{3}{4}$.

We say that $\frac{1}{6}$ is a multiplicative inverse of 6, because $6 \times \frac{1}{6} = 1$. In general:

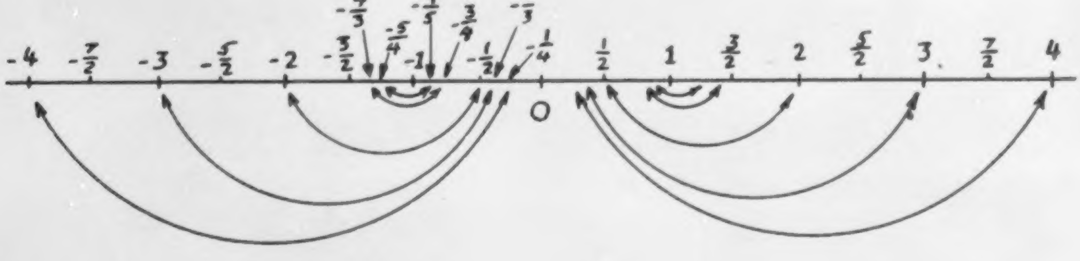
If c and d are real numbers such that

$$cd = 1,$$

then d is a multiplicative inverse of c.

If d is a multiplicative inverse of c, then is c a multiplicative inverse of d? Why?

We can observe something of the way these inverses behave by looking at them on the number line. On the diagram below, some sumbers and their inverses under multiplication are joined by double arrows. How can you test to see that these pairs of numbers really are multiplicative inverses? Can you visualize the pattern of the double arrows if a great many more pairs of these inverses were similarly marked?



How about the number 0? With what number can 0 be paired? Is there a number b such that $0 \times b = 1$? What can you conclude about a multiplicative inverse of 0?

Exercises 7 - 4.

1. If b is a multiplicative inverse of a, what values for b do we obtain if a is larger than 1? What values of b do we obtain if a is between 0 and 1? What is a multiplicative inverse of 1?
2. What values for b do we get if a is less than -1? If $a < 0$ and $a > -1$? What is a multiplicative inverse of -1?
3. Find inverses under multiplication of the following numbers: $3, \frac{1}{2}, -3, -\frac{1}{2}, \frac{3}{4}, 7, \frac{5}{6}, -\frac{3}{7}, -7, \frac{3}{10}, \frac{1}{100}, -\frac{1}{100}, 0.45, -6.8$.
4. Draw a number line and mark off with double arrows the first six numbers given in problem 3 and their multiplicative inverses.
5. If you had done problem 4 for additive inverses instead of multiplicative, how would the pattern of double arrows differ?
6. For inverses under multiplication, what values of the inverse b do you obtain if a is positive? If a is negative?

7 - 5. The inverse under multiplication is unique.

The above examples suggest that every real number (with the exception of 0) has an inverse under multiplication. We have seen that the additive inverse of a number is unique. (What does "unique" mean?) Is the multiplicative inverse of a number also unique? In other words, does there exist only one multiplicative inverse of a given number?

We can see, for instance, that $4 \times \frac{1}{4} = 1$, but can we imagine any number b other than $\frac{1}{4}$ for which $4 \times b = 1$? If so, what is it?

While it seems fairly obvious that $\frac{1}{4}$ is the only multiplicative inverse of 4, we need to follow a general argument which proves conclusively that there is only one inverse under multiplication for any real number, except zero. It is not so much that we doubt this statement as it is that we welcome another opportunity to exercise our reasoning ability in a proof. It is customary to call a property which can be proved a theorem.

Theorem 7 - 1. For each non-zero real number a there is only one multiplicative inverse of a.

Proof: Let us assume that a multiplicative inverse of a is b; that is, $ab = 1$. If there is another inverse under multiplication, say x, such that $ax = 1$, then we have

$$ax = 1,$$

$$b(ax) = b, \quad \text{(Why?)}$$

$$(ab)x = b, \quad \text{by the associative and commutative properties of multiplication,}$$

$$(1)x = b, \quad \text{(Why?)}$$

$$x = b.$$

Thus this "other inverse" x is really the number b; so there is no different inverse.

We shall find it convenient to use the shorter name "reciprocal" for the multiplicative inverse, and we represent the reciprocal of a by the symbol "$\frac{1}{a}$". Thus, for every a except 0, $a \times \frac{1}{a} = 1$.

You probably noticed that for positive integers the symbol we chose for "reciprocal" is the familiar symbol for a fraction. Thus, the reciprocal of 5 is $\frac{1}{5}$. This certainly agrees with your former experience.

But now the reciprocal of $\frac{2}{3}$ is $\frac{1}{\frac{2}{3}}$; of -9 is $\frac{1}{9}$; of 6.73 is $\frac{1}{6.73}$. Do these symbols represent fractions? If so, they must agree with the meaning of a reciprocal. That is, since $\frac{1}{\frac{2}{3}}$ is the reciprocal of $\frac{2}{3}$ and $\frac{2}{3} \times \frac{3}{2} = 1$, it follows that $\frac{1}{\frac{2}{3}}$ and $\frac{3}{2}$ must be the same number; since $\frac{1}{9}$ is the reciprocal of -9 and since $-9 \times (-\frac{1}{9}) = 1$, $\frac{1}{-9}$ and $-\frac{1}{9}$ must be the same number. We shall be in a better position to continue this discussion after we consider division of real numbers.

What is the reciprocal of: 15, -8, $\frac{1}{5}$, $-\frac{1}{6}$, $\frac{5}{3}$, 0.3, $-\frac{3}{4}$?

Why did we exclude 0 from our definition of reciprocals? Suppose 0 did have a reciprocal. What could it be? What is the truth set of the sentence $(0)(b) = 1$? You concluded that 0 simply cannot have a reciprocal. Here we have an opportunity to demonstrate, for a rather simple example, a very powerful type of proof. This proof depends on the idea that, given a sentence, it is either true or it is false, but not both. If we can show that one of these is impossible, the other must be the case. And one way to show that a sentence is true is to show that assuming it is false leads to a contradiction; therefore, the sentence is not false, and, hence, must be true.

Theorem 7 - 2. The number 0 has no reciprocal.

Proof: Assume that the sentence of the theorem is false. Then 0 has a reciprocal, say a. This would mean that

$$0 \times a = 1.$$

Since the product of zero and any real number is zero, it follows that

$$0 = 1.$$

This is a contradiction (of what?). Thus our assumption that zero has a reciprocal is a false assumption, and it follows that zero has no reciprocal.

7 - 6. Properties of reciprocals. We should like now to see what we can discover and what we can prove about the way reciprocals behave.

In each of the following sets of numbers, find the reciprocals. What conclusion do you draw about reciprocals on examining the two sets?

$$\text{I: } 12, \ \frac{1}{8}, \ 150, \ 0.09, \ \frac{8}{9}$$

$$\text{II: } -5, \ -\frac{1}{3}, \ -700, \ -2.2, \ -\frac{5}{3}$$

Observation of reciprocals on the number line strengthens our belief that the following theorem is true.

> Theorem 7 - 3. The reciprocal of a positive number is positive, and the reciprocal of a negative number is negative.

Proof: The statement follows immediately from the definition, $a \times \frac{1}{a} = 1$, since the product of two numbers is positive if and only if both numbers are positive or both numbers are negative.

For each of the following numbers, find the reciprocal of the number; then find the reciprocal of that reciprocal. What conclusion is suggested?

$$-12,\ 80,\ \frac{19}{20},\ -\frac{1}{9},\ 1.6$$

> Theorem 7 - 4. The reciprocal of the reciprocal of a non-zero real number a is a.

Proof: Since $\frac{1}{\frac{1}{a}}$ is the reciprocal of $\frac{1}{a}$ by the definition of a reciprocal, it follows that $(\frac{1}{a})(\frac{1}{\frac{1}{a}}) = 1$. Similarly, since $\frac{1}{a}$ is the reciprocal of a, it follows that $(a)(\frac{1}{a}) = 1$, or, by the commutative property, $(\frac{1}{a})(a) = 1$. Compare $(\frac{1}{a})(\frac{1}{\frac{1}{a}}) = 1$ with $(\frac{1}{a})(a) = 1$.

We see that the number $\frac{1}{a}$ has reciprocals $\frac{1}{\frac{1}{a}}$ and a. Since any non-zero real number has only one reciprocal, it follows that

$\frac{1}{\frac{1}{a}} = a$, which is what we wanted to prove.

By now you have seen several proofs like this one. You may still be wondering why we are interested in them, what good they are, and how we are expected to do them.

One reason why we are interested in proofs is that they hold together the whole structure of mathematics---or of any logical subject. We have set up, early in the course, the basic properties of the real number system, such as the commutative, associative, and distributive laws, and we have pointed out that all of algebra can be based on a short set of such properties. The proofs supply the connections between those basic properties, and all the many ideas which grow out of them become the whole subject of algebra.

If then, we are going to appreciate fully what mathematics is like, we should begin to experience some of this connecting process in the chain of reasoning---we should do some proving, and not always be satisfied with a plausible explanation. It is true that some of the statements we are proving seem very obvious, and we wonder, quite justifiably, why we should bother to prove them. As we progress further in mathematics, there will be more ideas which are less obvious, for which there will be a real need of a proof. If we waited until those more advanced stages, however, to start proving, we would find it very difficult. During the more elementary stages we should have the experience of seeing what proofs are like and developing some feeling for the continuous chain of reasoning on which the whole structure of mathematics depends. For these reasons we are taking time to look closely at proofs of some rather obvious statements.

The ability to discover a method for proving a theorem is something which comes by seeing a variety of proofs, by learning to look for connecting links between something you know and something you want to prove, by thinking about the suggestions which are given to lead you into a proof.

Let us use a second proof of Theorem 7 - 4 as an example. We want to prove that the reciprocal of the reciprocal of a is a.

It helps to write down what we know. Here, since we want to talk about the reciprocal of a and also the reciprocal of the reciprocal of a, we can write, by the definition

$$(a)\left(\frac{1}{a}\right) = 1 \quad \text{and} \quad \left(\frac{1}{a}\right)\left(\frac{1}{\frac{1}{a}}\right) = 1.$$

We should also write down what we want to prove.

$$\frac{1}{\frac{1}{a}} = a.$$

We then look closely for a way to change the thing we know into the thing we want to prove. We may make a guess after finding only part of the proof and try it out to see whether the rest works. The fact that we want the number a on the right suggests multiplying one of our given equations by a. The pattern on the left suggests that it is the second of our given equations which should be multiplied by a. So we try it.

$a(\frac{1}{a} \cdot \frac{1}{\frac{1}{a}}) = a(1)$, multiplication property of equality, (Sometimes we use a dot "$\cdot$" to mean "$\times$".)

$(a \cdot \frac{1}{a})\frac{1}{\frac{1}{a}} = a$, associative property of multiplication,

$(1)\frac{1}{\frac{1}{a}} = a$, definition of reciprocal, $a \cdot \frac{1}{a} = 1$,

$\frac{1}{\frac{1}{a}} = a$, property of one.

Exercises 7 - 6a.

1. Find the reciprocals of the following numbers:

   $\frac{3}{4}$, 0.3, -0.3, 0.33, -0.33, 1, -1, $\sqrt{2}$, $a^2 + 1$, $\frac{1}{x^2 + 4}$, $y^2 + 1$.

2. For what real values of a do the following numbers have no reciprocals?

   $a - 1$, $a + 1$, $a^2 - 1$, $a(a + 1)$, $\frac{a}{a + 1}$, $a^2 + 1$, $\frac{1}{a^2 + 1}$

3. Consider the sentence

   $$(a - 3)(a + 1) = a - 3,$$

   which has the truth set $\{0,3\}$. If both sides of the sentence are multiplied by the reciprocal of $a - 3$, that is by $\frac{1}{a - 3}$, and some properties of real numbers are used (which properties?), we obtain

   $$a + 1 = 1.$$

   For $a = 3$, we have $3 + 1 = 1$, and this is clearly a false sentence. Why doesn't this new sentence have the same truth set as the original sentence?

4. What property of opposites corresponds to Theorem 7 - 3 ? What property of opposites corresponds to Theorem 7 - 4 ?

5. Consider three pairs of numbers: (1) $a = 2$, $b = 3$; (2) $a = 4$, $b = -5$; (3) $a = -4$, $b = -7$. Does the sentence $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$ hold true in all three cases?

6. Is the sentence $\frac{1}{a} > \frac{1}{b}$ true in all three cases of problem 5 ? Plot the reciprocals on the number line.

7. Is it true that if $a > b$ and a, b are positive, than $\frac{1}{b} > \frac{1}{a}$? Try this for some particular values of a and b.

8. Is it true that if $a > b$ and a, b are negative, then $\frac{1}{b} > \frac{1}{a}$? Substitute some particular values of a and b.

9. Could you tell immediately which reciprocal is greater than another if one of the numbers is positive and the other is negative? Plot on the number line.

10. If $a < b$, and a and b are both positive real numbers, prove that $\frac{1}{a} > \frac{1}{b}$. Hint: Multiply the inequality by $(\frac{1}{a} \cdot \frac{1}{b})$. Demonstrate the theorem on the number line.

11. Does the relation $\frac{1}{a} > \frac{1}{b}$ hold if $a < b$ and both a and b are negative? Prove it or disprove it.

12. Does the relation $\frac{1}{a} > \frac{1}{b}$ hold if $a < b$ and $a < 0$ and $b > 0$? Prove or disprove.

In problem 5 above you showed that for three particular pairs of numbers a and b, $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$.

In other words, the product of the reciprocals of these two numbers is the reciprocal of their product. How many times would we need to test this sentence for particular numbers in order to be sure that it is true for *all* real numbers except zero? Would 1,000,000 tests be enough? How would we know that the sentence would not be false for the 1,000,001th test?

We can often reach *probably* conclusions by observing what happens in a number of particular cases. We call this *inductive reasoning*. But no matter how many cases we observe, inductive reasoning alone cannot assure us that a statement is *always* true. Thus, the fact that 900 successive automobiles have all stopped at a particular red light is no proof that the next car will also stop.

While we cannot prove that $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$ is always true by inductive reasoning, we *can* prove it for all non-zero real numbers by reasoning as follows:

Theorem 7 - 5. For any non-zero real numbers a and b,

$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab} .$$

Proof: Since

$$(ab)(\frac{1}{a} \cdot \frac{1}{b}) = (ba)(\frac{1}{a} \cdot \frac{1}{b}) = b(a \cdot \frac{1}{a})\frac{1}{b} = (b \cdot 1)\frac{1}{b} = b \cdot \frac{1}{b} = 1,$$

it follows that $(\frac{1}{a} \cdot \frac{1}{b})$ is the reciprocal of (ab); that is,

$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab} .$$

Supply the reason for each step in the proof above. Can you suggest why we began this proof by starting to work on $(ab)(\frac{1}{a}\cdot\frac{1}{b})$?

Notice how closely the proof of Theorem 7 - 5 parallels the proof that the sum of the opposites of two numbers is the opposite of their sum. Remember how this result was proved:

$(a + b) + \big((-a) + (-b)\big) = \big(a + (-a)\big) + \big(b + (-b)\big) = 0$; hence, $(-a) + (-b) = -(a + b)$.

Exercises 7 - 6b.

1. What is the value of $87 \times (-9) \times 0 \times \frac{2}{3} \times 642$?
2. Is $8\cdot17 = 0$ a true sentence? Why?
3. If $n\cdot50 = 0$, what can you say about n?
4. If $p\cdot0 = 0$, what can you say about p?
5. If $p\cdot q = 0$, what can you say about p or q?
6. If $p\cdot q = 0$, and we know that $p > 10$, what can we say about q?

The idea suggested by the above exercises will be a very useful one, especially in finding truth sets of certain equations. We are able to prove the following theorem now by using the properties of reciprocals.

Theorem 7 - 6. For real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

Because of the "if and only if", we really must prove two theorems.

Proof: If $a = 0$ or $b = 0$, then $ab = 0$ by the multiplication property of 0. Thus, we have proved one part of the theorem.

To prove the other part of the theorem, note that either $a = 0$ or $a \neq 0$. If $a = 0$, the requirement that $a = 0$ or $b = 0$ is satisfied. Why?

If $a \neq 0$, then

$$\left(\frac{1}{a}\right)(ab) = \frac{1}{a}\cdot 0, \qquad \text{(Why?)}$$

$$\left(\frac{1}{a}\right)(ab) = 0, \qquad \text{(Why?)}$$

$$\left(\frac{1}{a}\cdot a\right)b = 0, \qquad \text{(Why?)}$$

$$(1)b = 0, \qquad \text{(Why?)}$$

$$b = 0.$$

Thus in this case also the requirement that $a = 0$ or $b = 0$ is satisfied; hence, we have proved the second part of the theorem.

Exercises 7 - 6c.

1. If a is between p and q, is $\frac{1}{a}$ between $\frac{1}{p}$ and $\frac{1}{q}$? Explain.
2. If $(x - 5)\cdot 7 = 0$, what must be true about 7 and $(x - 5)$? Can 7 be equal to 0? What about $(x - 5)$ then?
3. Explain how we know that the only value of y which will make $9 \times y \times 17 \times 3 = 0$ a true sentence is 0.

4. How can we, without just guessing, determine the truth set of the equation $(x - 8)(x - 3) = 0$?

5. Find the truth set of each of the following equations:

(a) $(x - 20)(x - 100) = 0$

(b) $(x + 6)(x + 9) = 0$

(c) $x(x - 4) = 0$

(d) $(3x - 5)(2x + 1) = 0$

(e) $(x - 1)(x - 2)(x - 3) = 0$

(f) $2(x - \frac{1}{2})(x + \frac{3}{4}) = 0$

(g) $(3x - 5)(2x + 1) = 0$

(h) $9\,|x - 6| = 0$

7 - 7. The two basic operations and the inverse of a number under these operations. In the last few chapters we have focused our attention on addition and multiplication and on the inverses under these two operations. These four concepts are basic to the real number system. Addition and multiplication have a number of properties by themselves, and one property connects addition with multiplication, namely the distributive property. All our work in algebraic simplification rests on these properties and on the various consequences of them which relate addition, multiplication, opposite, and reciprocal.

We have pointed out that the distributive property connects addition and multiplication. It is instructive to see whether such a relationship occurs for every combination of addition, multiplication, opposite, and reciprocal in pairs. Let us write down all possible combinations.

1. Addition and multiplication: The distributive property, $a(b + c) = ab + ac$.

2. Addition and opposite: We have proved that
$-(a + b) = -a + (-b)$.

3. Addition and reciprocal: We find that there is no simple relationship between $\frac{1}{a} + \frac{1}{b}$ and $\frac{1}{a + b}$. In fact, there are no real numbers at all for which these two expressions represent the same number. This unfortunate lack of relationship is considerable cause of trouble in algebra for students who unthinkingly assume that these expressions represent the same number.

4. Multiplication and opposite: We have proved that
$-(ab) = (-a)(b) = (a)(-b)$.

5. Multiplication and reciprocal: We have proved that
$\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$.

6. Opposite and reciprocal: $\frac{1}{(-a)} = -\left(\frac{1}{a}\right)$.
This last relation is a new one and should be proved. The proof may be obtained from (5) above by replacing b by -1. The proof is left to the students. (Hint: What is the reciprocal of -1?)

State (1), (2), (4), (5), (6) in words. Do you see any similarity in these properties? Explain.

7 - 8. Division. You will recall that we defined

subtraction of a number as addition of the opposite of the number:

$$a - b = a + (-b).$$

In other words, we defined subtraction in terms of addition and the inverse under addition.

Since division is related to multiplication in much the same way as subtraction is related to addition, we might expect to define division in terms of multiplication and the inverse under multiplication. This is exactly what we do.

> For any real numbers a and b $(b \neq 0)$, a divided by b means a multiplied by the reciprocal of b.

We shall indicate "a divided by b" by the symbol "$\frac{a}{b}$". Then the definition of division is:

$$\frac{a}{b} = a \cdot \frac{1}{b}, \quad (b \neq 0).$$

As in arithmetic, we shall call a the *numerator* or *dividend* of $\frac{a}{b}$, and b the *denominator* or *divisor* of $\frac{a}{b}$. The resulting number $\frac{a}{b}$ is called the quotient *of* a *by* b or the *ratio of* a *to* b. Sometimes $\frac{a}{b}$ is also written a/b. For example,

$$\frac{10}{2} = 10 \cdot \frac{1}{2} = 5\ ; \quad \frac{3}{\frac{1}{5}} = 3 \cdot 5 = 15; \quad \frac{3}{4}\Big/\frac{3}{2} = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}.$$

(Of course, there is also the symbol "÷" for division which you used in arithmetic. We still shall use it occasionally, but the form "$\frac{a}{b}$" is more convenient and will be used most of the time in algebra.)

Why in the definition of division did we make the restriction "b/ 0"? Shall we ever be able to divide by zero? Be on your guard against being forced into an impossible situation by inadvertently trying to divide by zero.

Let us see whether this definition of division agrees with the ideas about division which we already have in arithmetic. An elementary way to talk about division of 10 by 2 is to ask, "How many 2's are there in 10?" Since there are five 2's in 10, and there are fifteen $\frac{1}{5}$'s in 3, the examples given above are verified.

Another way to think of division is to say, "a divided by b means that number which multiplied by b gives a." Since $(-5)(2) = (-10)$, then $\frac{-10}{2} = -5$; since $(-\frac{1}{5})(-15) = 3$, then $\frac{3}{-15} = -\frac{1}{5}$.

Exercises 7 - 8a.

1. Think of the division $\frac{a}{b}$ as
   - (a) How many b's are there in a?
   - (b) What number multiplied by b gives a?
   - (c) a times the reciprocal of b.
   - (d) Compare your results in (a), (b), and (c).

(i) $\frac{28}{0.01}$ (ii) $\frac{\frac{3}{5}}{\frac{1}{5}}$ (iii) $\frac{8}{\frac{1}{4}}$ (iv) $\frac{2600}{1}$

2. For the following, do parts (b), (c), and (d) indicated above.

(i) $\frac{28}{-2}$ (iii) $\frac{-5x}{x}$ (v) $\frac{\frac{3}{8}}{\frac{5}{7}}$

(ii) $\frac{6a}{3}$ (iv) $\frac{-24}{-6}$ (vi) $\frac{a^2}{-a}$

(vii) The quotient of 84 by 4.

(viii) The numerator is 28 and the denominator is -4.

(ix) The ratio of -5000 to 200.

(x) The divisor is $\frac{2}{5}$ and the dividend is $\frac{6}{5}$.

(xi) The denominator is -4 and the numerator is 2.

(xii) $12 \div 17$

In problem 2(xii) you found that to indicate the division, you wrote $\frac{12}{17}$. When you asked, "What number multiplied by 17 gives 12?", the best way you could answer was to write the fraction $\frac{12}{17}$. When you multiplied 12 by the reciprocal of 17, you got $12 \cdot \frac{1}{17}$ and could not go further.

Here we have another example of using the same symbol, this time the bar between the 12 and the 17, with two different meanings, but still being able to operate without confusion. Since 12 divided by 17 gives the fraction $\frac{12}{17}$, it does not matter whether we interpret the bar as a symbol saying "divide" or as a symbol between the numerator and the denominator of a fraction. This, incidentally, explains why we use "numerator and denominator" interchangeably with "dividend and divisor."

Actually, we have a third meaning of the bar in $\frac{1}{17}$, where it indicates the reciprocal of 17. But since the reciprocal of 17, and the fraction $\frac{1}{17}$, and the result of dividing 1 by 17, are all the same number, we may use the bar with any one of these three meanings.

The agreement we find between the results in parts (b) and (c) of Exercises 7 - 9a, suggests the following theorem:

Theorem 7 - 7. If $b \neq 0$, then $a = cb$ if and only if $\frac{a}{b} = c$.

This amounts to saying that "a divided by b is the number which multiplied by b gives a".

Again, in order to prove a theorem involving "if and only if" we must prove two things. First, we must show that if $\frac{a}{b} = c$ $(b \neq 0)$, then $a = cb$. The fact that we want to obtain cb on the right suggests starting the proof by multiplying both sides of $\frac{a}{b} = c$ by b.

Proof: If $\frac{a}{b} = c$ $(b \neq 0)$, then $a \cdot \frac{1}{b} = c$,

$$(a \cdot \frac{1}{b})b = cb,$$

$$a(\frac{1}{b} \cdot b) = cb,$$

$$a \cdot 1 = cb,$$
$$a = cb.$$

Second, we must show that if $a = cb$ $(b \neq 0)$, then $\frac{a}{b} = c$. This time, the fact that we do not want b on the right suggests starting the proof by multiplying both sides of $a = cb$ by $\frac{1}{b}$, which is posiible, since $b \neq 0$.

Proof: If $a = cb$ $(b \neq 0)$, then
$$a \cdot \frac{1}{b} = (cb)\frac{1}{b},$$
$$a \cdot \frac{1}{b} = c(b \cdot \frac{1}{b}),$$
$$a \cdot \frac{1}{b} = c \cdot 1,$$
$$a \cdot \frac{1}{b} = c,$$
$$\frac{a}{b} = c.$$

Supply the reason for each step of the above proofs.

The second part of this theorem supports our customary method of checking division by multiplying the quotient by the divisor.

Exercises 7 - 8b.

1. In the following problems perform the indicated divisions and check by multiplying the quotient by the divisor.

   (a) $\frac{2500}{-2}$

   (b) $\frac{-\frac{2}{3}}{3}$

   (c) $\frac{35p}{7p}$

   (d) $\frac{3\sqrt{5}}{3}$

   (e) $\frac{-45}{5}$

   (f) $\frac{-\frac{5}{8}}{-\frac{7}{8}}$

   (g) $\frac{-976}{-976}$

   (h) $\frac{9\pi}{3}$

   (i) $\frac{-200}{-50}$

   (j) $\frac{15a}{-3}$

   (k) $\frac{0}{48}$

   (l) $\frac{360}{2\pi}$

(m) $\frac{12}{\frac{1}{3}}$ (n) $\frac{-14}{0.1}$ (o) $\frac{93m}{-93}$ (p) $\frac{14\sqrt{2}}{2\sqrt{2}}$

2. Comment on $\frac{28}{0}$.

3. When dividing a positive number by a negative number, is the quotient positive or is it negative? What if we divide a negative number by a positive number? What if we divide a negative number by a negative number?

4. Prove that a quotient is positive if its dividend and divisor are both positive or both negative, and is negative if one is positive and the other is negative.

5. Show that if the quotient of two real numbers is positive, the product of the numbers also is positive, and if the quotient is negative, the product is negative.

6. Find the truth set of each of the following equations:

(a) $6y = 42$

(b) $2x = -70$

(c) $-5z = -20 + 5$

(d) $3x = 7 + 7x$

(e) $\frac{1}{5}|y| = 5 - 10$

(f) $\frac{a}{3} = -21 + 3$

(g) $\frac{3}{4}b = 1 - \frac{2}{3}$

(h) $38x = 0$

(i) $-\frac{5}{8}m = \frac{3}{7} + |-\frac{1}{7}| - \frac{m}{8} + \frac{m}{2}$

7. In problem 6, can you suggest more than one way of finding the required truth sets without guessing?

8. If it takes $\frac{2}{3}$ of a pound of sugar to make one cake, how many pounds of sugar are needed for 35 cakes for a banquet?

9. If six times a number is decreased by 5, the result is -37. Find the number.

10. If two-thirds of a number is added to 32, the result if 38. What is the number?

11. On a 20% discount sale, a chair is marked $30.00. What was the price of the chair before the sale?

12. One-half of a number is 3 more than one-sixth of the same number. What is the number?

13. Mary bought 15 three-cent stamps and some four-cent stamps. If she paid $1.80 for all the stamps, was she charged the correct amount?

14. John has 50 coins which are nickels, pennies, and dimes. He has four more dimes than pennies, and six more nickels than dimes. How many of each kind of coin has he? How much money does he have?

15. John, who is saving his money for a bicycle, said, "When I have one dollar more than three times the amount I now have, I will have enough money for my bicycle." If the bicycle costs $76, how much money does John have now?

16. Two trains leave New York at the same time; one travels north at 60 m.p.h. and the other south at 40 m.p.h. After how many hours will they be 125 miles apart?

17. A plane which flies at a average speed of 200 m.p.h. (when no wind is blowing) is held back by a head wind and takes $3\frac{1}{2}$ hours to complete a flight of 630 miles. What is the average speed of the wind?

18. The sum of three successive positive integers is 108. Find the integers.

19. The sum of two successive positive integers is less than 25. Find the integers.

20. Find two consecutive even integers whose sum is 46.

21. Find two consecutive odd positive integers whose sum is less than or equal to 83.

22. A merchant made a mixture of 150 lb. of tea worth $109.50 by mixing tea worth $1.25 a pound with tea worth 65 cents a pound. How many pounds of each kind did he use?

23. How many quarts of permanent antifreeze must be added to 3 gal. of a 10% solution to make it a 25% solution?

7 - 9. **Rational numbers.** We find that many numbers can be expressed as the quotient of two integers. (What is an integer?) Thus, $-5 = \frac{-10}{2}$, $3.87 = \frac{387}{100}$.

$\frac{2}{3}$ and $\frac{7}{8}$ are already written as quotients of integers.

We quite naturally ask whether *every* real number can be written as an integer divided by an integer. Suppose we try a few.

Exercises 7 - 9a.

In each of the following, try to express the number as the quotient of two integers. Is it possible to do this in more than one way?

1. -3
2. $4\frac{2}{3}$
3. $\frac{7}{11}$
4. $-\frac{1}{4}$
5. 66.4
6. $\frac{8}{12}$
7. $\sqrt{9}$
8. 4000
9. $\sqrt{2}$
10. -0.006
11. $-\frac{0.86}{0.99}$
12. π

You should be puzzled by problems 9 and 12 and should *not* have succeeded in writing a quotient of integers for these numbers. Later we shall prove that $\sqrt{2}$ *cannot* be written as the quotient of two integers. In fact, there are many such numbers which are not quotient of any integers.

> We shall use the name *rational* numbers for those numbers which can be expressed as the quotient of two integers. All other real numbers (such as $\sqrt{2}$, $\sqrt{3}$, π, $\sqrt[3]{7}$, and others which you have not yet met) we shall call *irrational numbers*.

Exercises 7 - 9b.

Consider the following sets of rational numbers of the form $\frac{a}{b}$ $(b \neq 0)$ where a and b are integers.

A: The set of all integers $(b = 1)$.

B: The set of all odd integers $(b = 1, a = \text{odd integer})$.

C: The set of all rational numbers having $b = 10$.

D: The set of all rational numbers having $b = 5$.

E: The set of all rational numbers having $a = 1$.

1. Graph a portion of each of the sets A to F on separate number lines.
2. For each of the above sets, decide whether the set is closed under addition. Explain.
3. For each of the above sets, decide whether the set is closed under multiplication. Explain.

4. Is set A a subset of any of the other sets? If so, which ones? Explain. Answer this same question for sets B, C, D, and E.
5. Describe the set which consists of all the numbers which are in both set C and set D. (This set is called the intersection of C and D.) Describe the intersection of A and E. Of A and D. Of B and C. Of C and E.
6. Describe the set which consists of all the numbers which are in either set B or set E. (This set is called the union of B and E.) Describe the union of C and D. Of A and D.
7. Is the set of all rational numbers closed under addition? Under subtraction? Under multiplication? Under division (excluding division by 0)?

7 - 10. Fractions. In arithmetic a fraction is the quotient of two counting numbers; so you see that the rational numbers include the fractions, as that word is used in arithmetic. Now we shall use the word "fraction" in a broader sense.

> Any quotient $\frac{a}{b}$, where a and b are any algebraic phrases ($b \neq 0$), will be called a fraction.

For example, $\frac{a+d}{2}$, $\frac{\sqrt{277}}{3}$, $\frac{a^2}{b^2+c^2}$, $\frac{7x^2-5x+13}{x-2}$, $\frac{\sqrt{x+y}}{x^2+y^2}$, $\frac{\pi}{2}$, $\frac{\frac{3}{5}}{\frac{1}{3}}$, will all be called fractions.

We now are able to describe the behavior of these fractions under various operations. Thus we shall see how the structure of algebra extends into a larger variety of expressions, and we also shall develop more algebraic techniques.

Let us first consider multiplication of fractions. In arithmetic you wrote "$\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$". Since fractions are quotients and we find quotients in terms of reciprocals, we shall first examine products of reciprocals, such as $\frac{1}{3} \times \frac{1}{7}$. Then we shall be able to prove quickly that we multiply fractions involving real numbers just as you did in arithmetic.

For products of reciprocals, we already have a theorem (Theorem 7 - 5) which states that, for all non-zero a and b,

$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab} .$$

Now apply the definition of division, $\frac{x}{y} = x \cdot \frac{1}{y}$, and the following theorem is readily proved.

Theorem 7 - 8. For real numbers a, b, c, d ($b \neq 0$, $d \neq 0$),

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} .$$

Proof:

$$\begin{aligned} \frac{a}{b} \cdot \frac{c}{d} &= (a \cdot \frac{1}{b})(c \cdot \frac{1}{d}) && \text{(Why?)} \\ &= (ac)(\frac{1}{b} \cdot \frac{1}{d}) && \text{(Why?)} \\ &= ac \cdot \frac{1}{bd} && \text{(Why?)} \\ &= \frac{ac}{bd} . && \text{(Why?)} \end{aligned}$$

For example, $\frac{7}{8} \cdot \frac{5}{11} = \frac{35}{88}$; $\frac{s}{3} \cdot \frac{t}{5} = \frac{st}{15}$;

$$\frac{2}{3} \cdot \frac{x + y}{w} = \frac{2(x + y)}{3w} = \frac{2x + 2y}{3w}$$

Theorem 7 - 8 tells us how to multiply one fraction by another. In order to see how to multiply a number by a fraction, we need only to let $b = 1$ in Theorem 7 - 8, and we obtain

$$a\left(\frac{c}{d}\right) = \frac{ac}{d}.$$

For example, $(-5)\left(\frac{7}{8}\right) = \frac{-35}{8}$; $3\left(\frac{x + y}{2}\right) = \frac{3x + 3y}{2}$.

Exercises 7 - 10a. Perform the indicated operations:

1. $\frac{3}{8} \cdot \frac{7}{2}$
2. $(-2) \cdot \frac{5}{9}$
3. $(-\frac{1}{4})(-\frac{5}{2})$
4. $\frac{1}{a} \cdot \frac{1}{a}$
5. $\frac{x}{3} \cdot \frac{x}{4}$
6. $\frac{5}{2} + \frac{3}{4}$
7. $10 \cdot \frac{19}{5}$
8. $(4a^2)(\frac{a}{3})$
9. $m \cdot \frac{3}{y}$
10. $\frac{3}{4}(x + 2)$
11. $\frac{a + 16}{2} \cdot \frac{b}{2}$
12. $\frac{a - 3}{5} \cdot \frac{a - 2}{2}$
13. $\frac{b}{x} \cdot \frac{b + 1}{x}$
14. $\frac{a}{c} \cdot \frac{a}{c}$
15. $\frac{y + 2}{4} \cdot \frac{y + 4}{2}$
16. Is every rational number a fraction? Is every fraction a rational number? Explain.

How can we find the truth set of the equation

$$\frac{2y}{3} + \frac{1}{2} = \frac{3y}{4} \ ?$$

One procedure is as follows: If there is a number y for which the sentence is true, then by the addition property of equality,

$$\frac{2y}{3} + \left(-\frac{3y}{4}\right) = -\frac{1}{2}. \qquad \text{(What was added?)}$$

Using the property of 1 and the distributive property, we may write

$$\frac{8y}{12} + \left(-\frac{9y}{12}\right) = -\frac{1}{2}$$

$$\frac{-y}{12} = -\frac{1}{2}$$

Now, by the multiplication property of equality,

$$y = 6 .$$

(Explain how each step was obtained by the property mentioned.) Now we find by checking that 6 is the solution of the equation.

But there is an easier way to find this truth set. Let us try to rid the equation of fractions in one operation. By what number should we multiply both sides of the equation to do this? Consider the fractions:

$$\frac{2}{3}, \qquad \frac{1}{2}, \qquad \frac{3}{4} .$$

Since we want only integers, what number when multiplied by each of these three fractions will yield an integer in each case? You will, of course, choose 12. Why? Would 24 do just as well?

Now let us multiply both sides of

$$\frac{2y}{3} + \frac{1}{2} = \frac{3y}{4}$$

by 12, giving the equation

$$8y + 6 = 9y .$$

Then it is easy to continue:

$$8y + (-9y) = -6 , \qquad \text{(Why?)}$$

$$-y = -6 , \qquad \text{(Why?)}$$

$$y = 6 . \qquad \text{(Why?)}$$

Express in your own words a procedure for ridding an equation of fractions.

Exercises 7 - 10b. Find the truth set of each of the following:

1. $\frac{x}{5} + 3 = \frac{2}{3}$
2. $\frac{7}{8} - \frac{1}{4}z = \frac{1}{2}$
3. $\frac{3}{5}x - \frac{1}{2} = \frac{8}{15}x$
4. $\frac{y}{2} + \frac{1}{3}y = \frac{2y}{5}$
5. $\frac{x}{7} + \frac{1}{2} > \frac{9}{14}$
6. $\frac{1}{9} + \frac{|y|}{6} \geq \frac{5}{12}$
7. $\frac{3}{4}x - \frac{2}{3} = \frac{1}{4} + \frac{5}{6}x$
8. $\frac{3}{7} + \frac{5}{4}y = \frac{3}{14} + \frac{3}{2}y + \frac{1}{7}$
9. $\frac{1}{3}z + \frac{1}{3} = 2 + \frac{1}{2}z$
10. $4y + 7 = y - 3$
11. $\frac{-3}{4} + \frac{4}{5}u = \frac{11}{8} + u - \frac{5}{8}$
12. $-\frac{1}{5} + \frac{5}{8}x = \frac{2}{3} + \frac{1}{2}x + \frac{1}{8}x$
13. $\frac{5}{16} - \frac{1}{5}|y| < \frac{3}{4} - \frac{7}{16}$
14. $\frac{3}{4} - |z - 1| = \frac{1}{4}$
15. $\frac{2}{3} > \frac{3}{4} + |z - 2|$

We saw earlier that a fraction, such as $\frac{2}{3}$, is often called the ratio of 2 to 3, or the ratio $\frac{2}{3}$. We also call a sentence in the form

$$\frac{a}{b} = \frac{c}{d}$$

a proportion. It is read "a, b, c, d are in proportion." These words are convenient when we are using division to show the relative size of two numbers. Since a ratio is a fraction, and a proportion is a simple sentence involving two fractions, these two words are just names for things with which we are already familiar.

Example: Two partners in a firm are to divide the profits in the ratio $\frac{3}{5}$. If the man receiving the larger share receives $8550, how much does the other partner receive?

If the smaller share is p dollars, then $\frac{p}{8550} = \frac{3}{5}$. If there is a number p such that the sentence is true, then

$$p = \frac{3}{5} \cdot 8550$$

$$p \quad 5130$$

If $p = 5130$, then $\frac{p}{8550} = \frac{5130}{8550} = \frac{3}{5} \times \frac{1710}{1710} = \frac{3}{5}$.

Hence, the smaller share is $5130.

Notice how saying that the shares are in the ratio $\frac{3}{5}$ leads naturally to writing the proportion $\frac{p}{8550} = \frac{3}{5}$.

Exercises 7 - 10c.

1. In a certain school the ratio of boys to girls was $\frac{7}{6}$. If there were 2600 students in the school, how many girls were there?
2. In a shipment of 800 radios, $\frac{1}{20}$ of the radios were defective. What is the ratio of defective radios to non-defective radios in the shipment?
3. The ratio of faculty to students in a college is $\frac{2}{19}$. If there are 1197 students, how many faculty members are there?
4. If two numbers are in the ratio $\frac{5}{9}$, explain why we may represent those numbers as $5x$ and $9x$. What are the numbers if $x = 7$? If $x = 100$? If their sum is 210?
5. Prove that if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.
6. Prove that if $ad = bc$ and $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$.
7. Show, using the properties of one, that the proportion $\frac{6}{15} = \frac{2}{5}$ is true.

8. Assuming that the proportion $\frac{x}{y} = \frac{p}{q}$ is true, use problems 5 and 6 to find seven other true proportions. For example, if $\frac{x}{y} = \frac{p}{q}$, then $xq = yp$ or $qx = yp$.

Hence, by problem 6, $\frac{q}{y} = \frac{p}{x}$.

7 - 11. Property of one. You have long known in arithmetic how to simplify $\frac{20}{45}$ to $\frac{4}{9}$. How do we know that these two fractions are equal? Does it help to write $\frac{20}{45}$ as $\frac{4}{9} \cdot \frac{5}{5}$? What is the value of $\frac{4}{9} \cdot 1$? Do you see how the number 1 is involved in this simplification?

Show similarly how the property of 1 explains why $\frac{2700}{200} = \frac{27}{2}$.

In order to extablish this use of 1 for all real numbers, we prove the following theorem.

Theorem 7 - 9. For any non-zero real number a, $\frac{a}{a} = 1$.

Proof: $\frac{a}{a} = a \cdot \frac{1}{a}$, (Why?)

$\frac{a}{a} = 1$. (Why?)

In order to simplify fractions, we shall use Theorem 7 - 9 and the property of 1 that $n \cdot 1 = n$.

Thus $$\frac{35xy}{14x} = \frac{5y}{2} \cdot \frac{7x}{7x} = \frac{5y}{2} \cdot 1 = \frac{5y}{2}$$

Have you noticed in this example how much use you make of factors? The reason why you can write $\frac{35xy}{14x}$ as $\frac{5y}{2} \cdot \frac{7x}{7x}$ is that you know that 5 and 7 and x and y are all factors of the numerator, and 2 and 7 and x are factors of the denominator. We then use the commutative and associative properties to arrange these factors so that a fraction of the form $\frac{a}{a}$ is obtained.

$$\frac{5\cdot 7\cdot x\cdot y}{2\cdot 7\cdot x} = \frac{(5\cdot y)(7\cdot x)}{(2)(7\cdot x)} = \frac{5y}{2}\cdot\frac{7x}{7x} = \frac{5y}{2}.$$

Exercises 7 - 11a.

In the following problems, perform the indicated operations and simplify the results, showing the use of the properties of 1.

1. $\frac{55}{121}$
2. $\frac{6y}{2y}$
3. $\frac{5p^2}{10p}$
4. $12\cdot\frac{3}{4}$
5. $(-\frac{7}{8})(-\frac{2}{3})$
6. $\frac{4b}{20}$
7. $\frac{6n^2}{7n}$
8. $\frac{26aw}{26aw}$
9. $\frac{130y}{10y^2}$
10. $\frac{2a}{b}\cdot\frac{3b}{a}$
11. $(-6x)\cdot\frac{5x}{4}$
12. $(\frac{3g}{7})(\frac{-14a}{g^2})$
13. $\frac{6(a + b)}{11(a + b)}$
14. $\frac{x - y}{-(x - y)}$
15. $\frac{3}{p + q}\cdot\frac{p + q}{-27}$
16. $\frac{3}{4}$

We saw in the above exercises how we are able to simplify fractions if there are factors which are the same in the numerator and denominator. On the other hand there are some situations in which it is convenient to increase the number of factors in a fraction; again we can use the properties of one to supply the factors we need.

For instance, how can we compare $\frac{34}{35}$ with $\frac{41}{42}$ to see which is greater? It is easy to compare $\frac{27}{29}$ and $\frac{31}{29}$ because their denominators are the same. If we could make the denominators of $\frac{34}{35}$ and $\frac{41}{42}$ the same without changing the value of each fraction, we could make our comparison easily. In the following series of steps, discuss why we perform each operation and how we are permitted to perform it.

$$\frac{34}{35} = \frac{34}{5\cdot 7},$$

$$\frac{41}{42} = \frac{41}{6:7},$$

$$\frac{34}{35} = \frac{34}{5\cdot 7}\cdot\frac{6}{6} = \frac{204}{5\cdot 6\cdot 7},$$

$$\frac{41}{42} = \frac{41}{6\cdot 7}\cdot\frac{5}{5} = \frac{205}{5\cdot 6\cdot 7}.$$

Which number is greater?

As another example of the use of the property of 1, consider the following division of fractions:

$$\frac{\frac{-3a^2}{5x}}{\frac{-2a}{3x^2}} = \frac{\frac{-3a^2}{5x}}{\frac{-2a}{3x^2}}\cdot\frac{15x^2}{15x^2} = \frac{\left(\frac{-3a^2}{5x}\right)(15x^2)}{\left(\frac{-2a}{3x^2}\right)(15x^2)}$$

$$= \frac{(-3a^2)(3x)}{(-2a)(5)} = \frac{(9ax)(-a)}{(10)(-a)} = \frac{9ax}{10}.$$

Why did we multiply by $\frac{15x^2}{15x^2}$? What led us to choose $\frac{15x^2}{15x^2}$?

Can you suggest in what other situation we need to supply more factors in the denominator of a fraction?

Exercises 7 - 11b.

1. Find a fraction equal to the given fraction with the indicated denominator, showing clearly your use of the properties of 1. For example, if we want a fraction with

denominator 8 and equal to $\frac{5}{2}$, we write $\frac{5}{2} \cdot \frac{4}{4} = \frac{20}{8}$.

(a) $\frac{7}{12}$ with denominator 36.

(b) $\frac{4a}{3b}$ with denominator 24ab.

(c) $\frac{-3}{5}$ with denominator 30.

(d) $\frac{a}{b}$ with denominator 7b.

(e) $\frac{2x}{3yw}$ with denominator $3y^2w^2$.

(f) $\frac{8}{-7}$ with denominator 21.

2. In each of the following pairs of numbers, determine which number is greater.

(a) $\frac{5}{9}$ and $\frac{6}{11}$

(b) $-\frac{13}{8}$ and $-\frac{11}{6}$

(c) $-\frac{67}{70}$ and $-\frac{57}{60}$

(d) $-\frac{42}{56}$ and $-\frac{36}{54}$

(e) $\frac{243}{43}$ and $\frac{242}{43}$

(f) $-\frac{48}{72}$ and $-\frac{36}{54}$

3. Arrange the following numbers in order of magnitude:

$$\frac{-7}{12}, \frac{6}{10}, \frac{11}{15}, \frac{-3}{4} .$$

4. Prove the theorem:

If $b \neq 0$ and $c \neq 0$, then $\frac{ac}{bc} = \frac{a}{b}$.

5. Prove the theorem:

If $a \neq 0$ and $b \neq 0$, then $\frac{1}{\frac{a}{b}} = \frac{b}{a}$.

6. Perform the indicated operations:

(a) $\frac{7}{9a} \cdot \frac{3a^2}{28}$

(b) $\frac{\frac{5}{9}}{\frac{10}{11}}$

(c) $\frac{1}{\frac{4x^2}{7}}$

(d) $9y \cdot \frac{3y}{7y}$

(e) $\frac{\frac{-2m^3}{3}}{3m^2}$

(f) $\frac{\frac{12ad}{-5x}}{\frac{-3ax}{10d}}$

(g) $\frac{13z}{5x} \cdot 5x$

(h) $\frac{|3|}{a} \cdot \frac{a}{|-3|}$

(i) $\frac{|x - 3|}{|3 - x|} \cdot \frac{4}{3}$

(j) $\frac{\frac{0}{2ay}}{\frac{6}{3by}}$

(k) $\frac{\frac{p + q}{2}}{\frac{p + q}{3}}$

(l) $\frac{\frac{-15n}{-2n^3}}{\frac{3n^2}{-4n}}$

(m) $\frac{17xy}{-3} \cdot \frac{-12x}{34y}$

(n) $\frac{a + 7}{5} \div \frac{a + 7}{4}$

(o) $\frac{x - 3}{2} \div \frac{3}{x - 2}$

7. Find the truth set of each of the following equations:

(a) $\frac{7}{8}y = \frac{3}{4}$

(b) $\frac{11}{5}x = -\frac{5}{2}$

(c) $-\frac{a}{9} = -3$

(d) $-\frac{3}{4}m = \frac{7}{2}$

(e) $\frac{x}{\frac{6}{7}} = \frac{6}{7}$

(f) $\frac{|y|}{\frac{1}{8}}$

(g) $-\frac{5}{6}w = -\frac{1}{6}$

(h) $\frac{3}{2}b + 5 = 8$

(i) $\frac{1}{3}x + 6 = 2$

8. Can you describe more than one way of finding the truth set in problem 7(a)?

9. Which is greater, $\frac{41}{71}$ or $\frac{41}{72}$? If the numerators of two fractions are the same and the denominators are different and all are positive numbers, how can you tell in general which is the greater fraction?

10. Which is greater for any positive real number x, $\frac{x}{x+5}$ or $\frac{x}{x+4}$?

7 - 12. Adding fractions. Explain each of the steps in the following:

$$\frac{8}{3} + \frac{-7}{3} = 8\left(\frac{1}{3}\right) + (-7)\left(\frac{1}{3}\right) = (8-7)\left(\frac{1}{3}\right) = \frac{1}{3} .$$

In general, if $\frac{a}{c}$ and $\frac{b}{c}$ are fractions (with the same denominator), their sum is

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} .$$

How shall we add fractions when their denominators are not the same? If we are going to calculate sums such as $\frac{3}{8} + \frac{5}{2}$, we must change the form of one or more of the fractions involved so that their denominators become the same without changing the value of any fraction.

$$(1)\quad \frac{3}{8} + \frac{5}{2} = \frac{3}{8} + \frac{5}{2}\cdot\frac{4}{4} = \frac{3}{8} + \frac{20}{8} = \frac{23}{8} .$$

$$(2)\quad \frac{1}{4} + \frac{5}{6} = \frac{1}{4}\cdot\frac{3}{3} + \frac{5}{6}\cdot\frac{2}{2} = \frac{3}{12} + \frac{10}{12} = \frac{13}{12} .$$

$$(3)\quad \frac{a+2b}{8} + \frac{2a-b}{6} = \frac{a+2b}{8}\cdot\frac{3}{3} + \frac{2a-b}{6}\cdot\frac{4}{4} .$$

$$= \frac{3a+6b}{24} + \frac{8a-4b}{24} = \frac{3a+6b+8a-4b}{24} = \frac{11a+2b}{24} .$$

In this third example, the smallest number which is a multiple of both 8 and 6 is 24.

For the present, inspection and a little thought will be sufficient to find that number which will best serve as the same denominator for all the fractions. In the next chapter we shall learn how, with the help of factors, we may find that number systematically in more complicated cases.

Exercises 7 - 12.

1. Perform the following additions and subtractions, using the form indicated in the examples above:

   (a) $\frac{5}{9} + \frac{2}{3}$ (d) $\frac{-3}{4} - \frac{3}{8}$ (g) $\frac{2}{3} + \frac{a}{2}$

   (b) $\frac{7}{10} - \frac{2}{5}$ (e) $\frac{4}{a} + \frac{5}{2a}$ (h) $\frac{5}{3x^2} - \frac{11}{6x^2}$

   (c) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4}$ (f) $\frac{4a}{7} - \frac{a}{35}$ (i) $\frac{x + 8}{10} + \frac{x - 4}{2}$

2. Find the truth set of each of the following equations:

   (a) $\frac{1}{3}a + \frac{7}{12} = \frac{3}{4}$ (e) $3|u| + \frac{4}{9} = \frac{3}{2}|u| - \frac{2}{21}$

   (b) $-\frac{2}{5}y - \frac{3}{4} = \frac{7}{10}$ (f) $-\frac{5}{6}y + 16 > -3y + \frac{1}{2}$

   (c) $\frac{2}{3}x + \frac{5}{6} = \frac{1}{9}$ (g) $-\frac{5}{12} + |u - 3| < -\frac{7}{24}$

   (d) $\frac{7}{24} + \frac{3}{5}x = \frac{4}{3} - \frac{2}{3}x$

3. Prove that $\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$ for real numbers a, b, and c ($c \neq 0$).

4. Prove that $\frac{a}{c} + \frac{b}{d} = \frac{ad + bc}{cd}$ for real numbers a, b, c, and d ($c \neq 0$, $d \neq 0$).

5. Why do we not need a separate theorem for subtracting fractions?

6. The sum of two numbers is 240, and one number is $\frac{3}{5}$ times the other. Find the two numbers.

7. The numerator of the fraction $\frac{4}{7}$ is increased by an amount x. The value of the resulting fraction is $\frac{27}{21}$. By what amount was the numerator increased?

8. $\frac{13}{24}$ of a number is 13 more than $\frac{1}{2}$ of the number. What is the number?

9. Joe is $\frac{1}{3}$ as old as his father. In 12 years he will be $\frac{1}{2}$ as old as his father then is. How old is Joe? His father?

10. The Yankee's on August 1 had won 48 games and lost 52. They have 54 games left on their schedule. Let us suppose that to win the pennant they must finish with a standing of at least .600. How many of their remaining games must they win? What is the highest standing they can get? The lowest?

11. The sum of two positive integers is 7 and their difference is 3. What are the numbers? What is the sum of the reciprocals of these numbers? What is the difference of the reciprocals?

12. (a) If it takes Joe 7 days to paint his house, what part of the job will he do in one day? How much in d days?
    (b) If it takes Bob 8 days to paint Joe's house, what part of the job would he do in one day? In d days?
    (c) If Bob and Joe work together what portion of the job would they do in one day? What portion in d days?
    (d) Referring to parts (a), (b), (c), translate the

following into an English sentence:

$$\frac{d}{7} + \frac{d}{8} = 1.$$

Solve this open sentence for d. What does d represent?

(e) What portion of the painting will Joe and Bob, working together, do in one day?

7 - 13. Complex fractions. If a fraction has one or more fractions in its numerator or denominator, it is called a complex fraction. Consider the example:

$$\frac{\frac{3}{2} + \frac{3}{4}}{\frac{1}{2} + \frac{7}{8}} = \frac{\frac{3}{2} + \frac{3}{4}}{\frac{1}{2} + \frac{7}{8}} \cdot \frac{8}{8} = \frac{12 + 6}{4 + 7} = \frac{18}{11}.$$

Observe that multiplying by 1 in the form $\frac{8}{8}$ eliminates fractions from both the numerator and the denominator. In this example a number of steps were done mentally. Find those steps and explain them, giving the properties used. Why did we choose $\frac{8}{8}$? How could you decide in general what number to use for this purpose? What number would you pick for this purpose? What number would you pick for this purpose for each of the following examples?

$$\frac{\frac{3}{7} + 5}{\frac{2}{7} - \frac{1}{3}} \qquad \frac{\frac{5}{2} + \frac{1}{3}}{\frac{3}{4} + \frac{1}{2}} \qquad \frac{\frac{5}{x} + \frac{6}{y}}{\frac{1}{x} - \frac{1}{y}}$$

Exercises 7 - 13. Perform the indicated operations in problems 1 - 19.

1. $\dfrac{\frac{2}{3} + \frac{1}{6}}{\frac{5}{6}}$

2. $\dfrac{2}{\frac{1}{8} + \frac{5}{12}}$

3. $\dfrac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{3} - \frac{1}{4}}$

4. $\dfrac{\frac{-3ax}{17}}{\frac{5x}{8}}$

5. $\dfrac{\frac{3}{a} - \frac{5}{2a}}{\frac{3}{b} + \frac{5}{2b}}$

6. $\dfrac{\frac{a - b}{2}}{\frac{a - b}{-4}}$

7. $\dfrac{\frac{6x}{5} - \frac{x}{4}}{\frac{x}{2} + \frac{2x}{5}}$

8. $\dfrac{\frac{x + 8}{9}}{\frac{3}{x + 2}}$

9. $\dfrac{\frac{1}{y + 1}}{\frac{y + 1}{3y}}$

10. $(\frac{3}{8} - \frac{1}{2}) + (\frac{5}{4} + \frac{1}{16})$

11. $(\frac{1}{9} + \frac{2}{3})(\frac{\text{[illegible]}}{3} - \frac{5}{6})$

12. $\frac{a + 6}{a - 6} \cdot \frac{a + 2}{a - 2}$

13. $\frac{a^2 - 3}{9} \div \frac{a^2 - 3}{9}$

14. $\dfrac{1 - \frac{2}{x} + \frac{1}{x^2}}{1 - \frac{1}{x^2}}$

15. $\dfrac{\frac{a}{b} + \frac{b}{a}}{\frac{1}{a} + \frac{1}{b}}$

16. $\frac{x - 2}{3 - x} \cdot \frac{x - 3}{2 - x}$

17. $\frac{x - 2}{3 - x} \div \frac{2 - x}{x - 3}$

18. $\dfrac{\frac{3}{x - 1} + 1}{\frac{-5}{x + 1}}$

19. $\dfrac{\frac{2}{a - 1} + \frac{1}{a + 1}}{\frac{1}{a - 1} \quad \frac{2}{a + 1}}$

Find the truth set of each of the equations in problems 20 - 25.

20. $\dfrac{\frac{a-1}{2}}{\frac{3}{4}} = 1$

21. $\dfrac{2x}{\frac{1}{8} + \frac{5}{12}} = 1$

22. $\dfrac{2x}{\frac{1}{8} + \frac{5}{12}} = 0$

23. $\dfrac{|y - 2|}{\frac{3}{4} + \frac{4}{3}}$

24. $\dfrac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{3} - \frac{1}{4}} > |x|$

25. $\dfrac{\frac{6x}{5} - \frac{x}{4}}{\frac{1}{2} + \frac{4}{5}} = x$

26. Show how the division can be performed in problems 6, 8, 9 by using the definition of division: $\frac{a}{b} = a \cdot \frac{1}{b}$.

7 - 14. Review Exercises.

1. Write the first step in using the distributive property to expand $(3x + 5)(2x - 3)$.

2. Why is the following sentence true?

$$\big(25 + (-10)\big) + (-25) = \big(25 + (-25)\big) + (-10).$$

3. Show that $\frac{3}{8} < \frac{9}{20}$ and $\frac{9}{20} < \frac{7}{15}$ are true sentences. Then tell why you know immediately that $\frac{3}{8} < \frac{7}{15}$ is true.

4. Each edge of a square is made twice as long. How much has the perimeter been increased? How much has the area been increased?

5. A man distributes \$24 between his children in amounts proportioned to their ages. The older is 7, and the younger 5. How much should each receive?

6. In a class of 10 pupils the average grade was 72. The students with the two highest grades, 94 and 98, were transferred to another class, and the teacher decided to find the average of the grades of the 8 remaining students. What was the new average?

7. Given the set S of all the even integers, (positive and negative and zero) which of five operations, (1) addition (2) subtraction (3) multiplication (4) division (5) average, applied to every pair of the elements of S, will give only elements of S? Describe your conclusions in terms of "closure".

8. A haberdasher sold two shirts for \$3.75 each. On the first he lost 25% of the cost and on the second he gained 25%

of the cost. How much did he gain or lose, or did he break even on the two sales?

9. Find the average of the numbers

$$\frac{x+3}{x},\ \frac{x-3}{x},\ \frac{x+k}{x},\ \frac{x-k}{x},\ \text{where } x \neq 0.$$

10. A man travels 360 miles due west at a rate of 3 minutes per mile and returns by plane at a rate of 3 miles per minute. What was his total traveling time? What was his average rate of speed for the entire trip?

11. A set of ten numbers has a sum t. If each number is increased by 4, then multiplied by 3, and then decreased by 4 the new total will be how much? If you had twenty numbers instead of ten and the same conditions, what would be the new total?

12. If $17 + a = 0$, what property of real numbers tells us at once that the sentence will be true for $a = -17$?

13. Which of the following name real numbers?

(a) $3 \cdot \frac{1}{8}$ (b) $3 \cdot \frac{1}{0}$ (c) $3 \cdot 0$ (d) $\frac{0}{0}$ (e) $\frac{|0|}{7}$ (f) $5 \div 0$

(g) $\left|\frac{1}{0}\right|$. For each part write either the simplest name for the number or the reason why it is not a number.

14. A rectangular swimming pool 25 yard's long and x yards wide has a sidewalk of seven-foot width built about it. If the outer edge of the walk forms a rectangle, write an open phrase for the number of feet in the perimeter of that outer edge.

15. Find the truth set of the open sentence

$$2|2| - 2|-12| = |2 - 12| .$$

16. Find the truth set of the open sentence

$$x + 6 - 5x < (-5)(-2) .$$

17. Draw the graph of the compound sentence

$$(|x - 6| = 4 \text{ or } |x| = 2) .$$

18. $A = \{0, |-1|\}$ and $B = \{-1, 0, 1\}$.

(a) Under which of the operations: addition, subtraction multiplication, division, is set A closed? set B?

(b) If C is the set of numbers obtained by squaring elements belonging either to set A or set B, enumerate set C. Is it a subset of A? of B?

19. Given the fraction $\frac{3x + 5}{2x - 7}$; what is the only value of x for which this is not a real number?

20. (a) A positive rational number is equal to $\frac{2}{3}$. If its numerator is less than 24, what can be said of its denominator?

(b) If the denominator is less than 24 what can be said of its numerator?

21. The product of two numbers is 2. If one of the numbers is less than 3, what is the other? If one is less than -3, what is the other?

22. Does division have the associative property? This is, is $(a \div b) \div c = a \div (b \div c)$? Give reasons for your answer.

23. Is division commutative? Give reasons for your answer.

24. For real numbers a, b, c show that if $ac = bc$ and $c \neq 0$,

then a = b.

25. If $\frac{1}{4}$ of a number increased by $\frac{1}{8}$ of the number is less than the number diminished by 25, what is the number?

26. Last year's tennis balls cost d dollars a dozen. This year the price if c cents per dozen higher than last year. What will half a dozen balls cost at the present rate?

27. A boy has 95 cents in nickels and dimes. If he has 12 coins, how many are nickels?

28. If $x = a + \frac{1}{a}$ and $a = \frac{1}{2}$, what is the value of $ax + a^2$?

29. William has five hours at his disposal. How far can he ride his bicycle into the woods at the rate of 8 miles per hour and return at the rate of 10 miles per hour?

30. A procedure sometimes used to save time in averaging large numbers is to guess at an average, average the differences, and add that average to your guess. Thus, if the numbers to be averaged—say your test scores—are 78, 80, 76, 72, 85, 70, 90, a reasonable guess for your average might be 80. We find how far each of our numbers is from 80.

$78 - 80 = -2$
$80 - 80 = 0$
$76 - 80 = -4$
$72 - 80 = -8$
$85 - 80 = 5$
$70 - 80 = -10$
$90 - 80 = 10$

The sum of the differences is -9. The average of the differences is $-\frac{9}{7}$. Adding this to 80 give $78\frac{5}{7}$ for the desired average. Can you explain wny this works?

31. The weights of a university football squad were posted as:

195, 205, 212, 201, 198, 232, 189, 178, 196, 204, 182. Find the average weight for the team by the method of differences explained in problem 23.

32. (Optional) Given the set {1, -1, j, -j} and the following multiplication table.

x	1	-1	j	-j
1	1	-1	j	-j
-1	-1	1	-j	j
j	j	-j	-1	1
-j	-j	j	1	-1

(a) Is the set closed under multiplication?

(b) Verify that this multiplication is commutative for the cases (-1, j), (j, -j) and (-1, -j).

(c) Verify that this multiplication is associative for the cases (-1, j, -j) and (1, -1, j).

(d) Is it true that $a \times 1 = a$, where a is any element of {1, -1, j, -j}.

(e) Find the reciprocal of each element in this set.

If x is an unspecified member of the set, find the truth sets of the following (make use of question (e)).

(f) $j \times x = 1$.

(g) $-j \times x = j$.

(h) $j^2 \times x = -1$.

(i) $j^3 \times x = -j$.

Chapter 8.

Factors, Exponents, Radicals.

8 - 1. Factors and divisibility. Once upon a time, there was a farmer whose total property amounted to 11 cows. This farmer had three sons, and when he died, he left a will which provided that $\frac{1}{2}$ his cows should be left to Charles, $\frac{1}{4}$ of his cows to Richard, and $\frac{1}{6}$ of his cows to Oscar. The boys quarreled greatly over this, because none of them wanted the non-integral pieces of bovine matter which the will seemed to require. As they were arguing, along the road came a stranger, leading a cow which he was taking to market. The three boys confided their problem to him, and the stranger replied: "That's simple. Just let me give you my cow, and then try it." The boys were delighted, for they now had 12 cows instead of 11. Charles took half of these, 6; Richard took his quarter, that is 3; and Oscar his sixth, namely 2 cows. The 11 cows which the father had left were now happily divided; the stranger took his own cow and went on his way.

So that you may not complain that the boys did not get exactly what the will provided, observe that each in fact got more, for $6 > \frac{11}{2}$, $3 > \frac{11}{4}$, and $2 > \frac{11}{6}$ (can you prove these inequalities?). However, there is something fishy about the problem, and it must be with the provisions of the will. What is it that made such an unusual solution possible?

This anecdote does have one mathematical conclusion at which we now want to look. For some reason, it was much easier to deal with 12 cows than with 11. And what was this reason? It was that 6 and 4 and 2 all divided into 12 exactly, while none of these, and indeed very little else, seems to divide into 11 exactly. And this is an important distinction between 12 and 11: there are many numbers which divide evenly into 12, but very few that divide into 11.

It's a bit clumsy to write "divide into" all the time, and so we shall use a more compact mathematical term for this. We shall say that 6 is a "factor" of 12 because $6 \times 2 = 12$; similarly, 4 is a factor of 12 (because $4 \times 3 = 12$), and so on. Is 3 also a factor of 12? Is 2?

The number 5, however, would not be a factor of 12, because we cannot find another integer such that 5 times that integer equals 12. Of course 1 and 12 are also factors of 12. Given any positive integer, 1 and the integer itself always divide that integer; because such factors are always present, they are not very interesting. So we shall call 2 and 3 and 4 and 6 proper factors of 12; these and 1 and 12 are all factors. The number 11, however, does not have any proper factors, because no positive integer other than 1 and 11 will divide 11 evenly. Now we are ready for a more precise definition of a proper factor:

The positive integer m is a proper factor of

> the positive integer n if $mq = n$, where q is a positive integer which equals neither 1 nor n.

Does it follow from this definition that m also can equal neither 1 nor n?

Try to write a similar definition of "factor" (without the "proper"). Since 3 is a factor of 18, then is $\frac{18}{3}$ a factor of 18? Is it true that if m is a factor of n, then $\frac{n}{m}$ is a factor of n? Is the same true for proper factors? How can you tell?

__Exercises__ 8 - 1a.

If the answer to the question is "Yes", write the number in factored form as in the definition. If the answer to the question is "No", justify in a similar way.

Sample: Is 5 a factor of 45? Yes, since $5 \times 9 = 45$.
Is 5 a factor of 46? No, since there is no integer q such that $5q = 46$.

1. Is 2 a factor of 24?
2. Is 3 a factor of 24?
3. Is 5 a factor of 24?
4. Is 6 a factor of 24?
5. Is 9 a factor of 24?
6. Is 13 a factor of 24?
7. Is 12 a factor of 24?
8. Is 24 a factor of 24?

9. Is 13 a factor of 91?

10. Is 30 a factor of 510?

11. Is 12 a factor of 204?

12. Is 10 a factor of 100,000?

13. Is 3 a factor of 151,821?

14. Is 6 a factor of 151,821?

15. Is 12 a factor of 187,326,648?

If any of the following numbers are factorable (i.e. have proper factors), find such a factor, and the product which equals the given number and uses this factor.

16. 85
17. 51
18. 52
19. 29
20. 93
21. 92
22. 37
23. 94
24. 55
25. 61
26. 23
27. 123
28. 57
29. 65
30. 122
31. 68
32. 95
33. 129
34. 141
35. 101

Let us now consider for a moment how you can tell whether 2 is a factor of a given number. Which of the numbers with which you just worked did have 2 as a factor? Is there an easy way to tell whether or not 2 is a factor of a number? Can you convince yourself that your answer is right?

Let us now look at the numbers 5 and 10. When is 5 a factor of some integer? You have probably known this for some time; every multiple of 5 ends in either a 5 or a 0, and everything that ends in a 5 or a 0 is a multiple of 5. Also every multiple of 10 ends in 0, and every number which ends in 0 is a multiple of 10. But we can now look at this in a slightly different way: a number has 10 as a factor if and only if it has both 2 and 5 as factors. Numbers which have 5 as a factor must end in 5 or 0, and numbers which have 2 as a factor must be even; hence, if a number is to have both 2 and 5 as a factor it must end in 0. Can you formulate what we have just said in terms of two sets and members common to both?

Exercises 8 - 1b.

Think about a test to check whether a number is divisible by 4, and also a test for divisibility by 3. The following examples should give you some real hints on the solutions - but don't be disappointed if a simple rule for 3 to be a factor of a number escapes you for the moment, for it is rather tricky.

1. Divisibility by 4: Which of the following numbers have 4 as a factor? 28, 128, 228, 528, 3028; 6, 106, 306, 806, 2006; 18, 118, 5618; 72, 572? Do you see the test? How many digits of the number do you have to consider?
2. Divisibility by 3: Which of the following numbers have 3 as a factor? 27, 207, 2007, 72, 702, 270; 16, 106, 601, 61, 1006. How about 36, 306, 351 (observe that $5 + 1 = 6$), 315, 513, 5129, 32122? Do you remember how divisibility by 9 was handled in problem 4 of Exercise 3 - 6, page 92? We wrote

$$\begin{aligned} 2357 &= 2(1000) + 3(100) + 5(10) + 7(1) \\ &= 2(999 + 1) + 3(99 + 1) + 5(9 + 1) + 7(1) \\ &= 2(999) + 3(99) + 5(9) + 2(1) + 3(1) + 5(1) + 7(1) \\ &= \big(2(111) + 3(11) + 5(1)\big) 9 + (2 + 3 + 5 + 7) \end{aligned}$$

$$= (222 + 33 + 5)9 + (2 + 3 + 5 + 7).$$

The expression $(222 + 33 + 5)9$ is divisible by 9; is it therefore divisible by 3? Remembering the rule for divisibility by 9, can you now formulate a rule for divisibility by 3? (Warning: If a number is divisible by 3, is it divisible by 9?)

3. If you know a test for both 2 and 3, what would be a test for 6?

(Optional) Certain modern electronic computing machines use the octal system of numeration for their internal symbolic codes. It is interesting to consider divisibility of a number written to the base eight.

4. The sequence of even numbers to the base 8 would be 2, 4, 6, 10, 12, 14, 16, 20,.... Propose a rule for divisibility by 2. Can you find an argument which would convince most people that your rule is correct? (Hint: Can you write any number to the base 8 as $n(8)$ + last digit?)
5. The numbers with 3 as a factor are the multiples of three. In the base 8 these are 3, 6, 11, 14, 17, 22, 25, 30. Is there a simple rule for divisibility by 3?
6. You can see that rules which were simple for the base 10 may not be simple for the base 8. Conversely, tests which were difficult to devise in base 10 may be easy in base 8. Devise a rule for divisibility of a number in the base 8 by 7. (Hint: Write out the multiples of 7, get an idea, and then try to find a convincing argument. Think similarly

to the way you would for 3 to the base 10.) Since an easy rule can be devised, would it be worthwhile to convert base ten numerals to base eight numerals in order to test for divisibility by 7? (There is no simple test for 7 with numbers to the usual base 10.) Would such a conversion save any time?

7. Devise a test for 4 to be a factor of a number written to the base 8.

8 - 2. Prime numbers. We have been talking of factors of positive integers over the positive integers, in the sense that when we write

$$mq = n$$

we accept only positive integers for m, n and q. We could, of course, if we wanted to, accept negative integers, or any rational numbers, or even any real numbers, as factors. But if you consider these possibilities for a moment, you will see that they do not add much to our understanding. If, for example, you permit negative integers as factors, do you really find anything new? How are the factors which involve negative integers related to those which involve positive integers only?

You get a different picture if you accept all rational numbers as possible factors of positive integers. The rational number $\frac{2}{7}$, for example, would be a factor of 13, in this extended sense, because $(\frac{2}{7})(\frac{91}{2}) = 13$. Can you think of any rational number, in fact, which would not be a factor of 13 in this sense? Is the situation any different if you permit factoring over arbitrary real numbers?

You see that if you try factoring positive integers over the rationals or over the reals, then everything becomes a factor of everything. Such a kind of factoring, therefore, would not add anything to our understanding of the structure of the real number system, and so we shall not consider it further. Only factoring over the positive integers tells us interesting things, and so when we speak of "factoring" a positive integer, we shall always mean over the positive integers.

Most of you are quite familiar with the set of positive integers since you have been working with this set since kindergarten or earlier. But have you even considered the fact that all you need to "generate" the whole set is the number 1 and addition? For if you add 1 and 1, you get 2; add 1 to 2, you get 3; add 1 to 3, you get 4; and so on. By adding 1 to each new number generated you will generate a new number. By continuing this process indefinitely you will have generated the entire set of positive integers.

Exercises 8 - 2a.

1. Generate a set of numbers using 2 and addition only. What is this set called? Which members of this set have 2 as a proper factor?
2. Generate a set of numbers using 3 and addition only. What is this set called? Which members of this set have 3 as a proper factor?

3. We have listed below a set of positive integers less than or equal to 100. Cross out the numbers in which 2 is a proper factor and write a 2 below each of these numbers.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

What is the first number after 2 which has not been crossed out? It should be 3. Now cross out all numbers which have 3 as a proper factor and write a 3 below each of the numbers. If a number has already been crossed out with a 2 do not cross it out again but skip it. What is the first number after three which has not been crossed out? Cross out numbers which have as proper factors the answer to the previous question. Continue the process. After the fifth step your picture should look like this.

1	2	3	$\cancel{4}_2$	5	$\cancel{6}_2$	7	$\cancel{8}_2$	$\cancel{9}_3$	$\cancel{10}_2$
11	$\cancel{12}_2$	13	$\cancel{14}_2$	$\cancel{15}_3$	$\cancel{16}_2$	17	$\cancel{18}_2$	19	$\cancel{20}_2$
$\cancel{21}_3$	$\cancel{22}_2$	23	$\cancel{24}_2$	$\cancel{25}_5$	$\cancel{26}_2$	$\cancel{27}_3$	$\cancel{28}_2$	29	$\cancel{30}_2$
31	$\cancel{32}_2$	$\cancel{33}_3$	$\cancel{34}_2$	$\cancel{35}_5$	$\cancel{36}_2$	37	$\cancel{38}_2$	$\cancel{39}_3$	$\cancel{40}_2$
41	$\cancel{42}_2$	43	$\cancel{44}_2$	$\cancel{45}_3$	$\cancel{46}_2$	47	$\cancel{48}_2$	$\cancel{49}_7$	$\cancel{50}_2$
$\cancel{51}_3$	$\cancel{52}_2$	53	$\cancel{54}_2$	$\cancel{55}_5$	$\cancel{56}_2$	$\cancel{57}_3$	$\cancel{58}_2$	59	$\cancel{60}_2$
61	$\cancel{62}_2$	$\cancel{63}_3$	$\cancel{64}_2$	$\cancel{65}_5$	$\cancel{66}_2$	67	$\cancel{68}_2$	$\cancel{69}_3$	$\cancel{70}_2$
71	$\cancel{72}_2$	73	$\cancel{74}_2$	$\cancel{75}_3$	$\cancel{76}_2$	$\cancel{77}_7$	$\cancel{78}_2$	79	$\cancel{80}_2$
$\cancel{81}_3$	$\cancel{82}_2$	83	$\cancel{84}_2$	$\cancel{85}_5$	$\cancel{86}_2$	$\cancel{87}_3$	$\cancel{88}_2$	89	$\cancel{90}_2$
$\cancel{91}_7$	$\cancel{92}_2$	$\cancel{93}_3$	$\cancel{94}_2$	$\cancel{95}_5$	$\cancel{96}_2$	97	$\cancel{98}_2$	$\cancel{99}_3$	$\cancel{100}_2$

Did the picture change from the 4th step to the 5th step? Why or why not? If you are having difficulty with this question perhaps it would help if you would consider the first number crossed out in each step. How far would the set of numbers have to be extended before the picture after the 5th step would be different from the picture after the 4th step?

In the set of 100 positive integers you have crossed out all the numbers which have proper factors. Thus the remaining numbers have no proper factors. We call each of these numbers a prime number.

> A prime number is a positive integer greater than 1 which has no proper factors.

Is it possible to find all the primes in the set of positive integers by the method we have just used (called the Sieve of Eratosthenes)? Is it possible to find all the primes less than some given positive integer by this method? What is the next prime after 97?

8 - 3. Prime factorization. Let us now return to the Sieve of Eratosthenes, which we have constructed, and see what else we can learn from it. Consider, for example, the number 63. It is crossed out, and hence 63 is not a prime. When did we cross 63 out? We see from the diagram that 63 was crossed out when we were working with 3. This means, if you stop and think about it, that 3 is the smallest prime factor of 63. (Actually, it follows from what we have just said that 3 is the smallest proper factor of 63, regardless of the "prime". Do you see why?)

Since 3 is the smallest prime factor of 63, let us divide it out. We obtain 21, and once again we can look in our chart to see if 21 is a prime. We find that it is not, and that in fact 3 is a factor of 21. Divide 21 by 3, and you obtain 7; if you look for 7 on the chart you find that it is not crossed out, so that 7 is a prime and can be factored no further. Now what is it that we have learned from all this? We have written 63 as 3 times 3 times 7; and the significance of this is that these factors of 63 are all primes. In other words, we have succeeded in writing 63 as a product of prime factors.

Let us try the same procedure again with 60. What prime were you considering when you crossed out 60? If you divide 60 by this prime, what do you obtain? Continue the process. What representation of 60 as a product of primes do you obtain?

Exercises 8 - 3a.

1. Using the Sieve of Eratosthenes, write each of the following numbers as a product of prime factors:

   84, 16, 37, 48, 50, 18, 96, 99, 78, 47, 12.

2. What positive integers have the following prime representations, respectively:

   $2 \times 2 \times 7$, $2 \times 3 \times 5$, 7×11, $2 \times 3 \times 3 \times 3$,
   7×7, $2 \times 2 \times 3 \times 3$?

A positive integer, you see, can be thought of as "made up" of a number of prime factors. Thus 63 is made up of two 3's and one 7; 60, if you did the above example correctly, is made up of two 2's, one 3, and one 5. We will have many uses for this "prime factorization", as it is called, of a positive integer. But now we face a problem: How do we do the same thing for a number which is not on our diagram? If you are asked for the prime factorization of 144, you might perhaps consider extending the diagram because, after all, it is not very far from 100 to 144. But suppose you are asked for 1764?

Maybe you can figure out a way to do the same thing without a picture of the sieve before you. What, after all, went on when you constructed the sieve? You first marked all numbers which were multiples of 2 with a "2"; the first number not marked was 3, and you proceeded to mark all numbers which were proper multiples of 3, and so on. Then came 5, and then 7. What were these numbers 2, 3, 5, 7, etc.? They were just the prime numbers. And so, what did happen to a number on the diagram? If 2 was a proper factor of it, it was crossed out when we were working with multiples of 2; if 2 was not a factor of it, but 3 was, then "3" was crossed out when we were working with multiples of 3, and so forth. If the number had no proper factors, i.e. was prime, it was not crossed out at all.

Let us now do the same thing without the sieve, say with 1764. We first have to try 2. Is 2 a factor of 1764? By the tests which we learned, the answer is, "Yes": 1764 = 2 × 882. So now let us try 882, as if we had the sieve before us. Is 2 a factor of 882? Yes, and 882 = 2 × 441. Now let us work on 441. Is 2 a factor of 441? No, it isn't; so if our sieve had gone as far as 441, this number would not have been crossed out when we considered multiples of 2. The next prime after 2 is 3, and so we must test next whether or not 441 is a multiple of 3. If you check 441 for divisibility by 3, either by the test in Exercises 8 - 1b or by actual division, you find that 3 *is* a factor of 441, so that it would have been crossed out in the sieve when we tested multiples of 3. We now obtain 441 as 3 × 147. There is no sense trying the factor 2 on 147, since if 2 were a factor of 147, it would also have been a factor of 441 (why?). But 3 divides 147, and we obtain 147 = 3 × 49. 49, in turn, is 7 × 7, and 7 is a prime number, so that the job is finished. To summarize: We have found that $1764 = 2 \times 2 \times 3 \times 3 \times 7 \times 7$, and this is the prime decomposition which we were looking for.

Of course all this writing is rather clumsy; a more compact way to do the whole job would be to write

$$\begin{array}{r|l} 1764 & 2 \\ 882 & 2 \\ 441 & 3 \\ 147 & 3 \\ 49 & 7 \\ 7 & \end{array}$$

Here the smallest prime factor at any stage is to the right of the line, and the quotient of the dividend by the smallest prime factor is underneath the dividend. The prime decomposition can be read off from the factors to the right of line, and the last remaining prime.

Exercises 8 - 3b.

1. What is the smallest prime factor of 115, of 135, of 321, of 484, of 539, of 121?
2. Find the prime decomposition of each of the following numbers: 98, 432, 258, 625, 180, 1024, 378, 729, 825, 576, 1098, 486, 3375, 3740, 1311, 5922, 1008, 5005, 444, 5159, 1455, 2324.

You may have noticed that we have been speaking of "the" prime factorization of a positive integer, as if we were sure that there was only one such factorization. Do you suppose that this is actually true? Can you give any convincing reasons why this should be the case? Does it help to think of a positive integer as "made up" of prime factors?

8 - 4. Lowest common denominator. One of the many places where the prime factorization of integers is important is in the addition and subtraction of rational numbers. It is easy to add fractions if their denominators are the same; you have also seen, in Chapter 7, some simple examples of adding two rational numbers with different denominators by finding a common denominator to which both could be transformed. We are now going to discover a systematic way of adding rational numbers with different denominators, a way which involves as little labor as possible.

To see just how much difference finding the lowest (meaning least) common denominator can make, consider the problem of simplifying

$$\frac{1}{4} - \frac{3}{10} - \frac{4}{45} + \frac{1}{6}.$$

One possible common denominator, of course, is the product of all the denominators. This would be $4 \times 10 \times 45 \times 6 = 10{,}800$. If all the arithmetic is performed correctly, we would find that the expression to be simplified equals

$$\frac{10 \times 45 \times 6 - 3 \times 4 \times 45 \times 6 - 4 \times 4 \times 10 \times 6 + 4 \times 10 \times 45}{10800}$$

$$= \frac{2700 - 3240 - 960 + 1800}{10800}.$$

This equals $\frac{300}{10800}$, or, finally, $\frac{1}{36}$.

You can imagine that one's chances of performing all this arithmetic correctly are pretty slim! Did we really have to get into such big numbers to do this problem? The fact that a factor 300 appeared in both numerator and denominator at the end makes us suspicious that perhaps we made the denominator too large from the very beginning.

What should the denominator have been? The denominator has to be a number which has 4 and 10 and 45 and 6 as factors. One such number is the product of all these, and this is the one we used before; but we feel certain that this was too large. How can we discover the smallest denominator we could have used?

We will get some ideas if we consider the prime factors of each of the denominators in the given problem:

$$4 = 2 \times 2,$$
$$10 = 2 \times 5,$$
$$45 = 3 \times 3 \times 5,$$
$$6 = 2 \times 3.$$

Since 4 must be a factor of the lowest common denominator (L.C.D.), this L.C.D. must, in its own prime factorization, contain at least two 2's. In order that 10 be a factor of the L.C.D., the latter's prime factorization must contain a 2 and a 5; we already have a 2 by the previous requirement that 4 be a factor, but we must also include a 5 now. To summarize what we have so far: in order that both 4 and 10 be factors of the L.C.D., the prime factorization of the L.C.D. must contain at least two 2's and one 5.

Next we must have 45 as a factor of the L.C.D. This means we have to supply two factors of 3 in addition to the two 2's and the 5 we already have; we don't need to supply another 5 because we already have one. Finally, to accomodate the factor 6, we need both a 2 and a 3 in the L.C.D. factorization, but we already have each of these.

Conclusion: The L.C.D. will have the prime factorization $2 \times 2 \times 3 \times 3 \times 5$. We need each of these factors, and any more would be superfluous. What superfluous factors did the denominator 10,800 contain?

Now that we have found the lowest common denominator, we can complete the problem by changing each of the rational numbers in our problem so that it has this denominator. What will $\frac{1}{4}$ become? One way to find this out is to multiply out our factored expression for the L.C.D.; we obtain $2 \times 2 \times 3 \times 3 \times 5 = 180$. Then 180 divided by 4 yields 45, so that, by the multiplication property of 1, $\frac{1}{4} = \frac{45}{180}$. An easier way to do the same thing, however, is to use the factored form of the L.C.D. and the factored form of 4 which we have found previously. 4 contains two 2's and nothing more, while the L.C.D. contains two 2's, two 3's and one 5. Thus, to change 4 into the L.C.D., we have to multiply by two 3's and one 5 to supply the missing factors.

$$\frac{1}{4} = \frac{1}{2 \times 2} = \frac{1}{2 \times 2} \times \frac{3 \times 3 \times 5}{3 \times 3 \times 5} = \frac{45}{2 \times 2 \times 3 \times 3 \times 5}.$$

Similarly,

$$\frac{3}{10} = \frac{3}{2 \times 5} = \frac{3}{2 \times 5} \times \frac{2 \times 3 \times 3}{2 \times 3 \times 3} = \frac{54}{2 \times 2 \times 3 \times 3 \times 5};$$

can you now do the same with $\frac{4}{45}$ and $\frac{1}{6}$? If you have completed the arithmetic correctly, you will now have

$$\frac{1}{4} - \frac{3}{10} - \frac{4}{45} + \frac{1}{6} = \frac{45 - 54 - 16 + 30}{2 \times 2 \times 3 \times 3 \times 5} = \frac{5}{2{\times}2{\times}3{\times}3{\times}5} = \frac{1}{2{\times}2{\times}3{\times}3} = \frac{1}{36}.$$

What is the advantage of this way of doing the problem? It is the avoidance of big numbers; the denominator is left in factored form until the very end, and you see that we never had to handle any number larger than 54. Compare this to the 10,800 of our first attempt!

Exercises 8 - 4a.

1. Find the following sums:

(a) $\frac{1}{85} + \frac{3}{51}$

(b) $-\frac{20}{57} - \frac{7}{95}$

(c) $\frac{5}{21} - \frac{3}{91}$

(d) $\frac{3x}{8} + \frac{5x}{36}$

(e) $\frac{1}{6} + \frac{3}{20} - \frac{2}{45}$

(f) $\frac{3k}{10} + \frac{2k}{28} - \frac{k}{56}$

(g) $\frac{3a}{5} + \frac{7a}{75} - \frac{5a}{63}$

(h) $\frac{x}{3} + \frac{5x}{8} - \frac{11}{70} + \frac{3}{20}$

2. Is it true that:

   (a) $\frac{8}{15} < \frac{13}{24}$?

   (b) $\frac{3}{16} < \frac{11}{64}$?

   (c) $\frac{14}{63} < \frac{6}{27}$?

3. Which is greater?

   (a) $\frac{1}{7}$ or $\frac{1}{2} - \frac{1}{3}$

   (b) $\frac{4}{15}$ or $\frac{7}{27}$

   (c) $\frac{5}{12}$ or $\frac{5}{13}$

4. You have learned in Chapter 5 that for any pair of numbers a and b, exactly one of the following must hold: $a > b$, $a = b$, or $a < b$. Put in the symbol for the correct relation for the pairs of numbers below:

   (a) $\frac{6}{27} \quad \frac{5}{28}$ (b) $\frac{2}{3} \quad \frac{5}{7}$ (c) $\frac{6}{16} \quad \frac{9}{24}$ (d) $(\frac{1}{2} + \frac{1}{3}) \quad (\frac{11}{12} - \frac{1}{13})$

5. A man is hired to sell suits at the AB Clothing Store. He is given the choice of being paid \$200 plus $\frac{1}{12}$ of his sales or a straight $\frac{1}{3}$ of sales. If he thinks he can sell \$600 worth of suits per month, which is the better choice? Suppose he could sell \$700 worth of suits, which is the better choice? \$1000? What should his sales be so that the offers are equal?

6. John and his brother Bob each receive an allowance of $1.00 per week. One week their father said, "I will pay each of you $1.00 as usual or I will pay you in cents any number less than 100 plus its largest prime proper factor. If you are not too lazy, you have much to gain." What number should they choose?

7. Suppose John's and Bob's father forgot to say proper factor. How much could the boys gain by their father's carelessness?

8 - 5. Some facts about factors. Suppose that you were looking for two factors of the number 72 with the property that their sum is 22. How would you go about finding them? One way, of course, would be to try all possible ways of factoring 72, and keep looking until you found a pair that met the condition. We are now going to see, however, that factors of numbers have properties which allow us to rule out many possibilities without actually trying them. The prime factorization of 72 is 2 × 2 × 2 × 3 × 3. The two factors of 72 which we are seeking must use up the three 2's and two 3's in the prime factorization of 72. Suppose three 2's were all in one factor, and no 2's in the other. Then one factor would be even, while the other factor would be odd, because it contained no 2's. But the sum of an even and an odd number is odd, and 22 is not odd. So this split of 72 won't work; we will have to split the three 2's between the two factors, and thus put two 2's in one factor, and one 2 in the other.

Next, let us look at the 3's. Do we split the two 3's, or do they both go into one of the two factors? Well, 22 does *not* have 3 as a factor; but if we were to split the two 3's in 72 between the two factors of 72, then each would have 3 as a factor, and then the sum would have 3 as a factor. The sum could certainly not be 22.

We have thus learned that the two factors of 72 which we want must be "put together" as follows: one factor contains two 2's while the other factor contains one 2; one factor contains both 3's, while the other contains no 3's. So there are only two possibilities; the two 3's go either with the one 2 or with the two 2's; but two 2's with two 3's makes 36, which is clearly too big, so that the two 3's go with the one 2 (making 18) and the other two 2's (making 4) form the other factor. Since $18 + 4 = 22$, we have our answer.

The kind of reasoning which we have just done depends on two ideas, namely: if 2 is a factor of one of two numbers, and 2 is a factor of their sum, then 2 is a factor of the other number; and if 3 is a factor of one of two numbers and 3 is *not* a factor of their sum, then 3 is *not* a factor of the other number.

Let us first prove a similar theorem.

Theorem 8 - 1. For positive integers b and c, if 2 is a factor of both b and c, then 2 is a factor of $(b + c)$.

Proof: $2q = b$, q an integer; because 2 is a factor of b;

$2p = c$, p an integer; because 2 is a factor of c.

Therefore,

$2q + 2p = b + c$, (Why?)

$2(q + p) = b + c$, distributive property.

Since $q + p$ is an integer,

then 2 is a factor of $(b + c)$.

State and prove a generalization of Theorem 8 - 1 in which the "2" is replaced by an arbitrary positive integer.

Now let us state a general result as a theorem.

Theorem 8 - 2. For positive integers a, b and c, if a is a factor of b and a is _not_ a factor of $(b + c)$, then a is _not_ a factor of c.

Proof: Assume a _is_ a factor of c; then a is a factor of both b and c and, hence, is a factor of $(b + c)$. (Why?) But this contradicts the given fact that a is _not_ a factor of $(b + c)$.

A third theorem useful in dealing with factors is

✓ Theorem 8 - 3. For positive integers a, b and c, if a is a factor of b and a is a factor of $(b + c)$, then a is a factor of c.

Exercises 8 - 5.

1. The prime factorization of 12 is 2 × 2 × 3. What two numbers, whose product is 12, have an even sum? odd sum? Can 3 be a factor of any possible sum?
2. The prime factorization of 36 is 2 × 2 × 3 × 3. Find two numbers whose product is 36 and whose sum will be divisible by 3 but not 2; divisible by 2 but not 3.
3. The prime factorization of 150 is 2 × 3 × 5 × 5. Find two numbers whose product is 150 and whose sum is even; divisible by 5; not divisible by 5.
4. Write the prime factorization of 18. Find two numbers made up of these prime factors whose product is 18 and whose sum is 9; 11.
5. Write the prime factorization of 288. Find two factors of 288 whose sum is 34.
6. Write the prime factorization of 972. Find two numbers made up of these prime factors whose product is 972 and whose sum is 247.
7. Find two factors of 216 whose sum is 37.
8. Find two numbers whose product is 270 and whose sum is 39.
9. The perimeter of a rectangular field is 68 feet and the area is 225 square feet. If the length and width are integers, find them.
10. Prove Theorem 8 - 3.
11. Show that if y is a positive integer, then y is a factor of $37(3y + y^2)$.

12. For what positive integer x is 3 a factor of $6 + 4x$? of $5 + 4x$?

13. If 3 boys shovel snow from sidewalks and charge 50¢ for a store and $1.50 for a house, how many walks of stores and houses must they shovel in order to split the money evenly? What if there were 4 boys?

14. Prove the theorems:

    (a) For positive integers a, b, c, if a is a factor of b and b is a factor of c, then a is a factor of c.

    (b) For positive integers a, b, c, d, if a is a factor of b, and c is a factor of d, then ac is a factor of bd.

    (c) For positive integers a, b, c, if a is a factor of b, and a is a factor of c, then a^2 is a factor of bc.

    (d) For positive integers a and b, if a is a factor of b, then a^2 is a factor of b^2.

15. Which theorem in problem 14 justifies the following:

    (a) 25 is a factor of (35) (15).

    (b) $(13)^2$ is a factor of (39) (26).

    (c) c^2 is a factor of $(5c)(9c)$ if c is a positive integer.

8 - 6. Introduction to exponents. We have seen that we can write a positive integer factored into its prime factors, so that, for example, $288 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3$. This notation is somewhat inconvenient because it is so lengthy; it would not be necessary to write the "2" five times if there were some way, more compact than $2 \times 2 \times 2 \times 2 \times 2$, of writing "five 2's

multiplied together".

As a first step in this direction, you already know that 3×3 can be written as 3^2. This is pronounced "3 squared"; the "3" indicated that we are going to multiply 3's together, and the "2" that we are going to multiply two of them. How would we write $2 \times 2 \times 2 \times 2 \times 2$ analogously? The number 288 can thus be written in factored form more compactly as $2^5 \times 3^2$.

In an expression of the form a^n, we need some way of describing the numbers involved. The "a", which indicates which number we are going to use as a factor several times, is called the "base"; the "n", which indicates how many of the factors "a" we are going to use, is called the "exponent". Thus a^n means a number consisting of n equal factors a; a^n is called a power, or more precisely, the n th power of a.

We can write

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ factors}}.$$

a^2 is read "a squared" or "a square".

a^3 is read "a cubed" or "a cube".

a^n is read "a to the n th power", or just "a to the n th".

Exercises 8 - 6a.

1. Can you guess how the words "squared" and "cubed" originated? If a side of a square is 5 inches long, how large, in square inches, is its area?
2. Find the prime factorization of each of the following numbers,

using exponents wherever appropriate. 64, 60, 80, 48, 128, 81, 49, 41, 32, 15, 27, 29, 56, 96, 243, 432, 512, 576, 625, 768, 686.

3. In describing the number a^n, what kind of number must n be? Must a be?

4. (Optional) The expression a^b can be thought of as defining a binary operation which, for any two positive integers a and b produces the number a^b. What does it mean to ask if this operation is commutative? Is it? What would it mean to ask whether or not this operation is associative?

Let us extend our notions about exponents. Since we know that the set of real numbers is closed under multiplication, it must be true that $a^3 \cdot a^2$ names a real number. Is there a simpler name? Since a^3 means that a is a factor three times and a^2 means that a is a factor twice, it follows that $a^3 \cdot a^2$ has a as a factor five times. That is,

$$a^3 \cdot a^2 = \overbrace{\underbrace{a \cdot a \cdot a}_{\text{3 factors}} \cdot \underbrace{a \cdot a}_{\text{2 factors}}}^{\text{5 factors}} = a^5$$

Write simple names for each of the following: $a^2 \cdot a^3$; $b^3 \cdot b^3$; $3^3 \cdot 3^4$; $(x^2)(x^5)$; $a^4 \cdot a^3 \cdot a^2$; $c^5 \cdot c^8$; $a^2 \cdot b^3$; $2^2 \cdot 3^3$. Notice that we sometimes write "$\cdot$" instead of "$\times$". Suppose we consider the number $a^m \cdot a^n$, where m and n are positive integers.

$$a^m \cdot a^n = \overbrace{\underbrace{a \times a \times a \times \cdots \times a}_{m \text{ factors}} \times \underbrace{a \times a \times a \times \cdots \times a}_{n \text{ factors}}}^{m + n \text{ factors}} = a^{m+n}$$

Does it seem reasonable, therefore, to say that $a^m \cdot a^n$ and a^{m+n} are names for the same number?

Have you noticed that we have been talking about a^2, a^3, a^5, a^7, etc., that is, forms of the type a^n, where n is a positive integer; but we have not mentioned a^1? Certainly, 1 is a positive integer and we should have a precise idea as to what a^1 means. Does it mean that a is a factor once? This intuitive notion seems reasonable but let us try to be more convincing. Certainly $a^2 \cdot a = a^3$. Now if we want to hold to our agreement that $a^m \cdot a^n$ and a^{m+n} name the same number, then $a^2 \cdot a^1 = a^{2+1} = a^3$. Thus, $a^2 \cdot a = a^2 \cdot a^1$ and it is clear that a must equal a^1. In this case, our intuitive notion is a good one.

Exercises 8 - 6b.

1. Write simpler names for the following:

   (a) $m^3 \cdot m^{11}$

   (b) $(x^3)(x^9)$

   (c) $(2x)(2^3x^2)$

   (d) $(27a)(3^4a^3)$

   (Hint: replace 27 by its prime factorization.)

   (e) $(16a^2)(32a^8)$

   (f) $(x^{2a})(x^a)$

   (g) $3^4 \cdot 3^2$

   (h) $3^4 \cdot 2^3$

   (i) $2^5 \cdot 3^2 \cdot 5 \cdot 2^2 \cdot 3^3 \cdot 5^2$

   (j) $2^a \cdot 2^{4a}$

   (k) $(3a^2b^3)(3^2ab^2)$

   (l) $(3k^2t)(3m^2t)$

2. Is $2^3 + 3^3 = 5^3$?
3. Is $2^3 \cdot 3^3 = 6^3$?
4. Is $2^3 + 3^3 = 6^3$?
5. Is $2^3 \cdot 3^3 = 6^6$?
6. Is $2^3 \cdot 3^3 = 6^9$?

7. Is $2^3 + 2^3 = 2^4$?

8. Is $2^3 + 2^3 = 2^6$?

8 - 7. Further laws of exponents. Now let us examine the fraction $\frac{a^5}{a^3}$. Is there a simpler name for this fraction? From the meaning of a^5 and a^3 it is evident that

$$\frac{a^5}{a^3} = \frac{a \times a \times a \times a \times a}{a \times a \times a} = a \times a \times \frac{a \times a \times a}{a \times a \times a} = a^2$$

Write simpler names for: $\frac{x^5}{x^2}$; $\frac{b^2}{b^3}$; $\frac{c^6}{c}$; $\frac{3^7}{3^2}$; $\frac{a^2}{a^2}$; $\frac{m^3}{m^2}$.

Can you generalize the results? Suppose we consider $\frac{a^5}{a^3}$ again, but reason in this way:

$$a^5 = a^3 \cdot a^{5-3} = a^3 \cdot a^2, \text{ because } a^m \cdot a^n = a^{m+n}.$$

Then

$$\frac{a^5}{a^3} = \frac{1}{a^3}(a^3 \cdot a^2) = (\frac{1}{a^3} \cdot a^3)a^2 = 1 \times a^2 = a^2.$$

Justify each step. Therefore, if $m > n$,

$$\frac{a^m}{a^n} = \frac{1}{a^n}(a^m) = \frac{1}{a^n}(a^n \cdot a^{m-n}) = (\frac{1}{a^n} . a^n)a^{m-n} = 1 \times a^{m-n} = a^{m-n} .$$

Justify each step. We specify that $m > n$ because we want $m - n$ to be a positive integer.

But suppose that $m = n$. For example, $\frac{a^3}{a^3} = 1$. (Why?)

Then

$$\frac{a^m}{a^n} = \frac{a^m}{a^m} = 1.$$

What if $m < n$? For example, $\frac{a^3}{a^5} = \frac{a \times a \times a}{a \times a \times a \times a \times a} = \frac{1}{a \times a} \times \frac{a \times a \times a}{a \times a \times a}$

$$= \frac{1}{a \times a} = \frac{1}{a^2}.$$

In general, if $m < n$, then

$$\frac{a^m}{a^n} = a^m\left(\frac{1}{a^n}\right) = a^m\left(\frac{1}{a^m \cdot a^{n-m}}\right) = a^m\left(\frac{1}{a^m} \cdot \frac{1}{a^{n-m}}\right) = \left(a^m \cdot \frac{1}{a^m}\right)\frac{1}{a^{n-m}}$$

$$= 1 \times \frac{1}{a^{n-m}} = \frac{1}{a^{n-m}}.$$

To summarize:

If $m > n$ then $\frac{a^m}{a^n} = a^{m-n}$.

If $m = n$ then $\frac{a^m}{a^n} = 1$.

If $m < n$ then $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.

Exercises 8 - 7a.

1. Write a simple name for each of the following: (We assume none of the variables takes on the value 0.)

(a) $\frac{m^3}{m^{11}}$

(b) $\frac{2x^4}{2^3x^2}$

(c) $\frac{(5^3x^2)(5x)}{(5^2x^6)}$

(d) $\frac{a^6b^6}{a\ b^2}$

(e) $a^7b^3c^2 \times a\ b\ c^3$

(f) $\frac{36a^2b^3}{8a^5b}$

(g) $\frac{288x^2y^3}{48x^6y^6}$

(h) $\frac{2^5 \cdot 5 \cdot 2^8 \cdot 5^3}{2^3 \cdot 5^5}$

(i) $\frac{54a^3b^{17}c}{153a^5b^{17}c}$

(j) $\frac{150h^2m^8}{225h^2t^8}$

(k) $\frac{81\ ax^2}{16a^4x^9}$

(l) $\frac{63a^2b\ a^1}{28a^3b^3}$

(m) $\frac{22a^3b^3c^3}{11a^3b^4c^2}$

(n) $\frac{24h^2c^3y}{16h^3bcy^2}$

(o) $\frac{49a^4bc^3}{14a^2b^2c^2}$

(p) $\frac{50c^4d^3y^2}{-5c^3dy^4}$

(q) $\frac{36x^2y^4z^2}{72x^3y\ z}$

2. Is $\frac{3^2}{2^2} = \frac{3}{2}$ true?

3. Is $\frac{6^3}{3^3} = 2$ true?

4. Is $\frac{3^4}{2^4} = \left(\frac{3}{2}\right)^4$ true?

5. Is $\left(\frac{4^3}{3^3}\right)\left(\frac{3}{4}\right)^3 = 1$ true?

6. Is $\frac{6^3}{3^3} = 2^3$ true?

7. Why must we be careful to avoid 0 values of the variables in problem 1?

8. (Optional) Having three rules for handling division is never as satisfactory as just one rule which will do the same job. It happens that it is possible to reduce all three rules to just one, namely: $\frac{a^m}{a^n} = a^{m-n}$, if we drop the condition $m > n$. Let us work some problems in two ways; first, using whichever rule of the last section is appropriate and second, using $\frac{a^m}{a^n} = a^{m-n}$. It is convenient to tabulate the results.

Complete the table

$\frac{a^7}{a^3} = a^{7-3} = a^4$	$\frac{a^7}{a^3} = a^{7-3} = a^4$
$\frac{a^3}{a^3} = \ ?$	$\frac{a^3}{a^3} = a^{3-3} = a^?$
$\frac{a^3}{a^5} = \frac{1}{a^{5-3}} = \frac{1}{a^?}$	$\frac{a^3}{a^5} = a^{3-5} = a^?$
$\frac{a^4}{a^4} = 1$	$\frac{a^4}{a^4} = a^{4-4} = a^?$
$\frac{a^2}{a^3} = \frac{1}{a}$	$\frac{a^2}{a^3} = \ ?$

We have extended notions of numbers in many instances before; can you now extend your notion of exponents? Examine the table carefully to answer the following questions:

$a^0 = \ ?$

$a^{-1} = \ ?$

$a^{-2} = \ ?$

Do zero and negative exponents make any sense in our definition of $a^n = a \cdot a \cdot a \cdots$ to n factors? Of course, it is senseless to think of a as a factor (-3) times. But does the following table make sense?

n	4	3	2	1	0	-1	-2	-3
2^n	16	8	4	2	1	$\frac{1}{2}$	?	?

So you see we can use just one rule for division, $\frac{a^m}{a^n} = a^{m-n}$, providing we define $a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$, where $a \neq 0$ and n is a positive integer. (Suppose n were allowed to be any integer; could you do away with the division rule altogether and use $a^m a^n = a^{m+n}$ for all cases?)

What is the meaning of $(ab)^3$? We know ab names a number, and we also know that a number cubed means that the number is a factor three times. Therefore, $(ab)^3$ must mean $(ab)(ab)(ab)$. By the commutative and associative properties of real numbers we know that

$$(ab)(ab)(ab) = (aaa)(bbb) = a^3b^3$$

Thus

$$(ab)^3 = a^3b^3$$

Write another name for $\frac{a}{b}^{3}$, using similar reasoning. Write another name for $(a^2b^3)^3$ using similar reasoning.

Exercises 8 - 7b.

1. Simplify (assuming no variable takes on the value 0):

(a) $\frac{(3a)^2}{12b^2a}$

(b) $\frac{(2^2 \cdot 3^3xy^4)^2}{2^4 \cdot 5x^2y^4}$

(c) $\frac{36a^2b}{24ab^2}$

(d) $\frac{256a}{288a^2b}$

(e) $\frac{(-3a)^2}{9}$

(f) $\frac{-3a^2}{9}$

(g) $\frac{(-2)^3a^2b^7}{8a^2b^5}$

(h) $\frac{(-2x^2yz)^3}{(-2x^2yz)^2}$

(i) $\frac{68a(b^2c^2)^2}{52(ab^2)^2c^2}$

(j) $\frac{8 \cdot 2^3m^2n}{97mn^2}$

(k) $\frac{-2^2a^2m}{128a^2m^3}$

(l) $\frac{(-2)^2a^2m}{128a^2m^3}$

(m) $\frac{(-a^3k)^2}{a^4(-k)^3}$

(n) $\frac{x^{2a}}{x^a}$

(o) $\frac{3}{4}^{\,2} \; \frac{28a^3}{45a}$

(p) $\left(\frac{ab}{c}\right)^2 \left(\frac{c^2}{a^3b}\right)$

(q) $\left(\frac{(mn)^2}{m}\right)\left(\frac{n^3}{m}\right)$

(r) $\left(\frac{2^3 \cdot 3^4}{5^2}\right)\left(\frac{5^3}{2^6 \cdot 3^2}\right)$

(s) $\left(\frac{63a^2}{243a^5}\right)\left(\frac{54a^7}{14a^4}\right)$

(t) $\frac{\frac{90(ab)^2}{16a^3}}{\frac{108ab^3}{81b}}$

(u) $$\dfrac{\dfrac{-30xy}{85x^2}}{\dfrac{150(-y)}{34z}}$$

(v) $$\dfrac{\left(\dfrac{2}{3}\right)^3 \left(\dfrac{ab}{c}\right)^2}{\dfrac{b^2}{16}}$$

(w) $$\dfrac{\dfrac{256a^8}{2^8a^8}}{\dfrac{25ab}{5(5a+10b)}}$$

(x) $$\dfrac{\dfrac{94(m^2n)^2}{34p}}{\dfrac{(p+p)}{4p^2}}$$

2. Is each of the following true? Give reasons for each answer.

(a) $\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2}$

(b) $\frac{2}{3} = \frac{2^2}{3^2}$

(c) $\left(\frac{5a}{7b}\right)^2 = \frac{5^2a^2}{7^2b^2}$

(d) $\frac{5a^2}{7b^2} = \frac{5^2a^2}{7^2b^2}$

(e) 3^3 is a factor of $(3^3 + 3^5)$

(f) 3^2 is a factor of $(6^2 + 9^2)$

(g) $(2x + 4y^2)$ is an even number, if x and y are positive integers.

3. Simplify:

(a) $\frac{11}{35a^2} + \frac{13}{25ab} - \frac{7}{5b^2}$

(b) $\frac{5}{2m^3} - \frac{2m^2}{mn^2} + \frac{m}{3n}$

(c) $\frac{11}{18x} - \frac{7}{30x^2} + \frac{9}{20x}$

(d) $\frac{3}{16} + \frac{3a}{64b} - \frac{7a}{54b^3}$

4. Prove: If a^2 is odd, then a is odd.

5. Prove: If a^2 is even then a is even.

6. Let a be 2, b be -2, c be 3, d be -3. Then determine the value of:

(a) $-2a^2b^2c^2$

(b) $(-2abc)^2$

(c) $\frac{-4a^4d}{6b^2a^3}$

(d) $\frac{a^2b^2c^4}{4a^3bc^2}$

(e) $\frac{-6a^{12}b^{16}c^{20}}{2a^{10}b^{18}c^{22}}$

(f) $\frac{(2a^2b^6d^3)^{10}}{(2a^2b^6d^3)^8}$

(g) $\frac{a^3+b^3}{a^3b^3}$

(h) $\frac{(ab)^3}{(a^2b^2)^3}$

(i) $\frac{(a+b)^3}{a^3+b^3}$

(j) $\frac{(a+b+c)^2}{a^2+b^2+c^2}$

8 - 8. Introduction to Radicals. Let us review for a moment the process of finding the square of a number. $6^2 = ?$, $(\frac{1}{2})^2 = ?$, $(300)^2 = ?$, $(-6)^2 = ?$, $(.8)^2 = ?$, $(x^2)^2 = ?$, $(-.8)^2 = ?$, $(-x)^2 = ?$, $(-2wa^3)^2 = ?$

Now, let us consider the same kind of question in the opposite direction. $(?)^2 = 49$, $(?)^2 = 100$, $(?)^2 = \frac{4}{9}$, $(?)^2 = .36$, $(?)^2 = 1$.

In the second group of questions we are finding, for example, a number whose square is 49. This is the inverse operation to squaring, and is called finding a square root. One number whose square is 49, and hence is a square root of 49, is certainly 7. Since it is also true that $(-7)^2 = 49$, it follows that -7 is also a square root of 49. Our notation and our terminology have to be chosen so that they will keep these two square roots distinct; people usually call the positive square root of a number b "the square root", and denote it by $\sqrt{b}$.

Let us now summarize this discussion.

> If b is a positive number, and $a^2 = b$, then a is a square root of b. If a is a square root of b, so is -a; the positive square root of b is denoted by $\sqrt{b}$, and is commonly called "the" square root of b. Another square root of b is then $-\sqrt{b}$.

We also have that $\sqrt{0} = 0$, in which case there is only one square root.

Exercises 8 - 8a.

1. Find:

(a) $\sqrt{4}$

(b) $-\sqrt{121}$

(c) $\sqrt{(-3)^2}$
(careful)

(d) $\sqrt{2.25}$

(e) $\sqrt{\frac{49}{9}}$

(f) $\sqrt{x^2}$
(even more careful)

(g) $\sqrt{.81}$

(h) $\sqrt{4y^2}$

(i) $\sqrt{4} + \sqrt{9} - \sqrt{25} + 3$

(j) $2\sqrt{\frac{49}{4}} - 3\sqrt{\frac{64}{9}}$

(k) $\sqrt{(2a+b)^2}$.

2. Is it possible that $\sqrt{x^2} + 2 = 1$ for some value of x? Explain.

3. Find the square root of $(2x-1)^2$ if

   (a) $x < \frac{1}{2}$, (b) $x > \frac{1}{2}$, (c) $x = \frac{1}{2}$.

4. Consider the following "proof" that all numbers are equal.

   If a and b are any real numbers, then

$$\begin{aligned} |a-b| &= |b-a|, \\ (a-b)^2 &= (b-a)^2, \\ a - b &= b - a, \\ 2a &= 2b, \\ a &= b. \end{aligned}$$

   Which step of this "proof" is faulty, and why?

   Find: $3^3 = ?$, $(-2)^3 = ?$, $(.4)^3 = ?$, $(x^2)^3 = ?$, $\left(\frac{2}{3}\right)^4 = ?$, $(-1)^5 = ?$, $(-3xy)^3 = ?$

Again, in the other direction, we can ask:

$(?)^3 = 1$, $(?)^3 = -1$, $(?)^3 = 1000$, $(?)^4 = 16$, $(?)^3 = -.027$.

Following the same procedure as before, we can say that a is a cube root of b if $a^3 = b$. We write $a = \sqrt[3]{b}$. Notice from the above examples that while we were not able to take square roots of negative numbers, since both negative and positive numbers have positive squares, it is perfectly possible to take cube roots of negative numbers, since the cube of a negative number is negative. On the other hand, we seem to be able to find only one number whose cube is 8, namely 2, while we have seen that numbers have two square roots when they have any at all. Within the framework of the real numbers, this is indeed correct; in the coming years, you will find that by extending the kinds of numbers we are willing to use, negative numbers will have square roots too, and all numbers will have three cube roots.

Exercises 8 - 8b.

1. Write a definition for *fourth roots*. For n th roots, where n is a positive integer. For what values of n do you think negative numbers will have real n th roots? How do you suppose the property of positive numbers of having two real square roots and one real cube root extends to n th roots?
2. What is the relation between $\sqrt[4]{16}$ and $\sqrt{4}$? Between $\sqrt[4]{10,000}$ and 100? Can you guess a relation between fourth roots and square roots that seems to be true?
3. Evaluate:
   - (a) $\sqrt[3]{729}$
   - (b) $\sqrt[3]{x^3}$
   - (c) $\sqrt[3]{-8y3}$
   - (d) $\sqrt[3]{.216}$
   - (e) $\sqrt[3]{-\frac{27}{8}}$
   - (f) $\sqrt[3]{(x-3y)^3}$
   - (g) $\sqrt[3]{64c^6}$
   - (h) $\sqrt[4]{81}$
   - (i) $\sqrt{\sqrt{81}}$.

8 - 9. *Radicals*. The symbol $\sqrt{\quad}$ is called the *radical sign*; an expression which consists of a phrase and a radical sign over it is called a *radical*.

Let us now return to square roots. Thus far, we have not attempted to take the square root of n unless we were able, with more or less difficulty, to recognize n as the square of some simple number or expression. (We call n a perfect square in this case.) Let us now consider the case of a square root which we cannot recognize immediately, such as, for example, $\sqrt{2}$. What kind of question do we want to ask about this? For instance, if we were faced with the expression $\sqrt{\frac{4}{9}} + 1$, we would not be happy to leave it in this form, for it can be simplified to read $\frac{2}{3} + 1$, or just $\frac{5}{3}$. But what if the expression were $\sqrt{2} + 1$? Should we expect to simplify this further, or have we gone as far as we can go? What does the answer to this question hinge upon? What do we mean, to "simplify", anyway?

Let us recall what happened in the case of $\sqrt{\frac{4}{9}} + 1$. We discovered that $\sqrt{\frac{4}{9}}$ was a rational number which we could combine with the rational number 1 and obtain the simpler expression $\frac{5}{3}$. Can we do something similar with $\sqrt{2} + 1$? The trouble is that, so far, we just do not know. We certainly do not know of any rational number whose square is 2, and yet we also do not know that there is no such number. Clearly, the time has come to settle this question once and for all.

Theorem 8 - 4. $\sqrt{2}$ is irrational.

Before we begin the proof of this theorem, let us think a bit of what we are up against. We want to prove that $\sqrt{2}$ is irrational, that is, that a number whose square is 2 cannot be rational. Now, how in the world does one prove that something does not have a certain property? Have we ever tried to do this before? Yes, we have. For example, we proved in Chapter 7 that 0 has no reciprocal. And how did we do this? We assumed that 0 did have a reciprocal, and showed that this assumption led to a contradiction. And if an assumption leads us to a contradiction, the assumption must be false; and if it is false that 0 has a reciprocal, then it has none. This reasoning worked in the other case, and so let us try it here.

Suppose that there were a rational number, say $\frac{a}{b}$, where a and b are positive integers, such that $\left(\frac{a}{b}\right)^2 = 2$. We can certainly insist that a and b have no common factor, for if they did, we could remove such a factor from the fraction $\frac{a}{b}$.

Now, if $$\left(\frac{a}{b}\right)^2 = 2,$$
then
$$\frac{a^2}{b^2} = 2, \qquad \text{(Why?)}$$
and
$$a^2 = 2b^2. \qquad \text{(Why?)}$$
This says that a^2 is an even number. But a itself is an integer, and hence must be either even or odd. If a were odd, then a^2 would also be odd (can you prove this?). But we know that a^2 is even. Thus a itself must be even. Then $a = 2c$, where c is another integer. (Why?) If we replace a by 2c in our last equation, we obtain
$$4c^2 = 2b^2,$$
and
$$2c^2 = b^2.$$

By the same argument which we just gave for a we know that b must now be even, since its square is even. So we have shown that both a and b must have been even. But a and b were chosen to have no common factor, and this certainly does not permit a and b to have the common factor 2. And now, finally, we have a contradiction; therefore, the assumption that $\sqrt{2}$ is rational has led us to a contradiction, and the assumption must have been false. Thus, $\sqrt{2}$ is irrational.

Notice, incidentally, an interesting difference between a proof by contradiction, such as we have just done, and most other proofs which you have seen during this course. In the usual proofs, there is a specific fact which you are trying to establish, and you proceed to work with whatever facts you are given and with the properties of the real numbers until the fact you are seeking is before you. You concentrate on creating the expression you desire from expressions which you have assumed to be true. In a proof by contradiction this is not exactly what you do. You add to your list of things with which you work the denial of what you want to prove, and then just keep deriving results until a contradiction appears. You don't know ahead of time just where this contradiction is coming from, but you keep working until you find one. You are thus not pointing specifically towards a fact which you are trying to prove, but you keep on gathering information until the information shows up something inconsistent. This inconsistency proves that you made a mistake in denying what you wanted to show, and thus what you wanted to show must have been true all along.

It is possible to establish in a similar way that the square root of any positive integer which is not a perfect square is irrational. Among the integers from 1 through 10, for example, this tells us that only 1, 4, and 9 have rational square roots, while the others have irrational square roots. Try to show that, for example, $\sqrt{3}$ is irrational.

Exercises 8 - 9a.

1. Since all integers are rational numbers, the fact that $\sqrt{2}$ is not an integer is actually included in Theorem 8 - 4. Try to show this directly.
2. (Optional). You learned in Chapter 1 that between any two points on the number line, there are always infinitely many points labelled with rational numbers. Do you think that between any two points on the number line there are infinitely many points whose coordinates are not only rational, but also perfect squares? On what part of the number line would it be worth trying for this result? Hint: The argument in Chapter 1 depended on the fact that, by averaging, you could always get a rational number between any two given rational numbers. Unfortunately, the average of two rational perfect squares is not necessarily a perfect square. Can you think of some other way of getting a rational perfect square between two of the same?

3. (Optional). Prove that $\sqrt[3]{2}$ is irrational.

8 - 10. Simplification of radicals. We observe that there can be only one positive number a which is a square root of n. For if there were a second such positive number, b, which is not the same as a, then either $a < b$ or $b < a$. (Why?) In these two cases, respectively, we would have $a^2 < b^2$ or $b^2 < a^2$, because of the multiplication property of order, and the squares of a and b could not both equal n.

Let us consider next the product of two square roots, say $\sqrt{2}$ and $\sqrt{3}$. Does this product equal some simpler expression? Whatever $\sqrt{2}\sqrt{3}$ equals, its square must (by what properties of multiplication?) equal 6. Thus $\sqrt{2}\sqrt{3}$ must be a square root of 6. Since we have just learned that there is only one positive number which is a square root of 6, it must be true that

$$\sqrt{2}\sqrt{3} = \sqrt{6}.$$

Exercises 8 - 10a.

1. Find a simpler expression for

(a) $\sqrt{5}\sqrt{6}$

(b) $\sqrt{2}\sqrt{7}$

(c) $\sqrt{3}\sqrt{11}\sqrt{2}$

(d) $\sqrt{2}\sqrt{x}$ (What restriction must be placed on x?)

(e) $\sqrt{z}\sqrt{3y}$

(f) $\sqrt{3}\sqrt{12}$

(g) $\sqrt{5}\sqrt{0}$

(h) $\sqrt{3}\sqrt{x^2}$.

2. Prove that for any two non-negative numbers a and b,

$$\sqrt{a}\sqrt{b} = \sqrt{ab}.$$

3. Is it true that for every real number a, $(\sqrt{a})^2 = a$?

Is it true for every non-negative number a?

We can use this fact about square roots found in problem 2 and separate a single square root into two. Using what we know about factors, we could, for example, write $\sqrt{48}$ as any of the following:

$\sqrt{6}\sqrt{8}$,
$\sqrt{4}\sqrt{12}$,
$\sqrt{3}\sqrt{16}$,
$\sqrt{2}\sqrt{24}$,
$\sqrt{3}\sqrt{8}\sqrt{2}$,

and many more. So far, however, this looks like just idle amusement. What is the point of trying any of these? Is any one of these forms of $\sqrt{48}$ simpler in some sense than any other, or, indeed, simpler than the original $\sqrt{48}$?

The first one, $\sqrt{6}\sqrt{8}$, certainly is no improvement, for it merely exchanges two radicals for one. The second expression, however, does have something in its favor, for 4 is a perfect square. Thus $\sqrt{4} = 2$, and we have shown that $\sqrt{48} = 2\sqrt{12}$. Since 12 is a smaller number than 48, this might well be considered an improvement. Now let us look at the next form: $\sqrt{48} = \sqrt{3}\sqrt{16}$. Once again, one of the radicals we have obtained contains a perfect square: $\sqrt{16} = 4$. By the commutative property of multiplication, we can now write $\sqrt{48} = 4\sqrt{3}$. Neither of the last two expressions in our list contains a perfect square under a radical sign; so they have nothing to offer us.

Let us concentrate on the two cases in which our factoring of 48 seems to have given us a simpler expression for 48: these are

$$\sqrt{48} = 2\sqrt{12},$$
$$\sqrt{48} = 4\sqrt{3}\ .$$

Can we tell which of these we might prefer? The difference between the two is that the first contains $\sqrt{12}$, while the second contains $\sqrt{3}$. But $\sqrt{12}$ can be written as $\sqrt{4}\sqrt{3}$, and 4 is yet another perfect square. Thus, in a very real sense, we are not finished when we have written $\sqrt{48} = 2\sqrt{12}$. On the other hand, 3 is a prime number, and since 3 contains no proper factors, it contains no proper perfect square factors either. Thus we can remove no more perfect squares from $\sqrt{3}$ and the simplification is finished. Furthermore, $\sqrt{3}$ is irrational and we can do no more with it. The simplest form we can find for $\sqrt{48}$, then, is $4\sqrt{3}$.

What have we really done? We have looked for various ways of factoring 48, and have picked those ways which made one of the factors a perfect square. Among these, the one we liked best was the one in which the perfect square was as large as possible. How could we find this largest perfect square factor without so much trial and error? Perhaps you remember one of our recent exercises: in the prime factorization of a perfect square, every prime factor appears an even number of times. To find the largest perfect square factor of a positive integer, therefore, look at the integer's prime factorization, and pick out as many factors as you can that appear in pairs. These are the ones which you want for your largest perfect square factor.

How would this have worked for $\sqrt{48}$? $48 = 2\times2\times2\times2\times3$, and the four two's can be considered as two pairs of two's. We have actually learned a neater way of saying the same thing: $48 = 2^4\times3$, and $2^4 = (2^2)^2$. Thus

$$\begin{aligned}\sqrt{48} &= \sqrt{2^4\times3},\\ &= \sqrt{2^4}\sqrt{3}, \quad \text{(Why?)}\\ &= 2^2\sqrt{3}, \quad \text{(Why?)}\\ &= 4\sqrt{3}\ .\end{aligned}$$

You may have wondered why, when we found an expression like $\sqrt{3}\sqrt{16}$, we preferred to write it as $4\sqrt{3}$ rather than $\sqrt{3}\times 4$. The reason is that we prefer not to write the extra multiplication symbol in the latter expression. Leave it out, you say, and write just $\sqrt{3}\ 4$? The trouble with this is that people are often not very careful how long they make their radical sign; the next thing you know, the expression will look like $\sqrt{34}$. It is to avoid this chance of confusion that we prefer to write $4\sqrt{3}$.

Exercises 8 - 10b.

1. Write in their simplest form the following:

Example. $\sqrt{54x^2}$. Solution:

$$\begin{aligned}\sqrt{54x^2} &= \sqrt{54}\sqrt{x^2},\\ &= |x|\sqrt{54}, \quad \text{(Why?)}\\ &= |x|\sqrt{2\cdot3^3},\\ &= |x|\sqrt{2\cdot3}\sqrt{3^2},\\ &= 3|x|\sqrt{6}.\end{aligned}$$

(a) $\sqrt{20}$
(b) $\sqrt{12}$
(c) $\sqrt{30}$
(d) $\sqrt{75}$
(e) $\sqrt{192}$
(f) $\sqrt{24}$
(g) $\sqrt{16}$
(h) $7\sqrt{63}$
(i) $\sqrt{28}$
(j) $3\sqrt{52}$
(k) $\sqrt{90}$
(l) $\sqrt{98}$
(m) $5\sqrt{42}$
(n) $\sqrt{18}$
(o) $16\sqrt{68}$
(p) $\sqrt{242}$
(q) $\sqrt{72}$
(r) $\sqrt{147}$
(s) $\sqrt{288}$
(t) $\sqrt{9 + 16}$
(u) $\sqrt[3]{16}$
(v) $\sqrt[3]{54}$
(w) $\sqrt[3]{192}$
(x) $\sqrt[3]{250}$
(y) $\sqrt{16 + 25}$

2. Find x, if x is a positive real number:
(a) $x^2 = 68$
(b) $x^2 = 56$
(c) $x^2 = 162$
(d) $x^3 = 56$

3. Simplify:
(a) $\sqrt{60x^2}$
(b) $\sqrt{32x^4}$
(c) $\sqrt[3]{81x^3}$
(d) $\sqrt{47x}$
(e) $\sqrt{46x^2}$
(f) $5\sqrt{600x}$
(g) $\sqrt{625x^2}$
(h) $\sqrt{169x^3}$
(i) $\sqrt{343x^5}$
(j) $\sqrt{9x^2+18x^3}$, for positive x
(k) $\sqrt{4x^2+8x}$, for positive x
(l) $\sqrt{x^4+x^2}$
(m) $\sqrt{x^2+9}$
(n) $\sqrt{500}$
(o) $\sqrt{5000}$

4. Simplify:

(a) $\sqrt{16}$

(b) $\sqrt[3]{16}$

(c) $\sqrt[4]{16}$

(d) $\sqrt[5]{16}$

(e) $\sqrt{108a^2}$

(f) $\sqrt[3]{27a^2}$

(g) $\sqrt[3]{-8a^3}$

(h) $\sqrt[6]{2^7}$

(i) $\sqrt{36a^2b^2}$

(j) $2x\sqrt{48x^2}$

(k) $\sqrt[3]{-27b^3}$

(l) $\sqrt{97x^2}$

(m) $\sqrt{1200}$

(n) $\sqrt{120}$

5. Find the truth sets of the following sentences:

(a) $2x^2 = 32$

(b) $\frac{1}{3}y^2 = 16$

(c) $\frac{3}{4}a^2 = 25$

(d) $2u^3 = 16$

(e) $\frac{1}{3u^3} = 9$

(f) $(n-1)^2 = 4$

6. Show that for positive real numbers a and b,

(a) $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$, (b) $\sqrt{a} = \sqrt{b}\,\frac{\sqrt{a}}{\sqrt{b}}$.

8 - 11. Simplification of radicals involving fractions.

We have seen what we would like to do with integers and various powers of variables under the radical sign, and what the goals of simplifying such expressions are. What do we do if we have a fraction inside the radical? One such case we have already handled; we had no hesitation about writing $\sqrt{\frac{4}{9}} = \frac{2}{3}$. Now suppose we were faced with $\sqrt{\frac{8}{9}}$? What can we do with this? We can write

$$\frac{8}{9} = \frac{4 \times 2}{9} = \frac{4}{9} \times 2.$$

(Why did we pick this particular factorization?) Then

$$\sqrt{\frac{8}{9}} = \sqrt{\frac{4}{9}}\sqrt{2} \text{ (Why?) } = \frac{2}{3}\sqrt{2}.$$

Since the only radical left is $\sqrt{2}$, which we know cannot be simplified, we are finished with this particular problem.

Consider another example:

$$\begin{aligned}\sqrt{\frac{3}{x^2}} &= \frac{\sqrt{3}}{\sqrt{x^2}}\\ &= \frac{\sqrt{3}}{|x|}\end{aligned}$$

For what set of values of x is the result sensible?

Exercises 8 - 11a.

1. Simplify:

(a) $\sqrt{\frac{16}{25}}$

(b) $\sqrt{\frac{4x^2}{9}}$

(c) $\sqrt{\frac{49}{4y^2}}$

(d) $\sqrt[3]{\frac{8}{27}}$

(e) $\sqrt{\frac{2}{9}}$

(f) $\sqrt{\frac{10x^2}{32}}$

(g) $\sqrt{\frac{6}{27a^2}}$

(h) $\sqrt{1 + \frac{24}{25}}$

(i) $\sqrt{\frac{24}{98}}$

(j) $\sqrt{6 + \frac{1}{4}}$

(k) $\sqrt{\frac{35}{4} + \frac{7}{9}}$

(l) $\sqrt{2 + \frac{2}{9}}$

(m) $\sqrt{\frac{15}{4} - \frac{5}{6} + \frac{5}{9}}$

(n) $\sqrt{\frac{11a^2}{18} + \frac{6a^2}{4} - \frac{7a^2}{6}}$

(o) $\sqrt{\frac{4}{15} - \frac{7}{75} + \frac{11}{45} - \frac{2}{9}}$

2. Perform the indicated operations:

(a) $\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}$

(b) $\frac{\sqrt{3a}}{\sqrt{a}} \cdot \frac{1}{\sqrt{a}}$ (What restriction on a?)

(c) $\frac{\sqrt{3x}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$

(d) $\frac{\sqrt{3}}{\sqrt{4}}$

We come now to the case of radicals containing fractions whose denominators are not perfect squares. What do we propose to do with $\sqrt{\frac{3}{5}}$, for example? We know that

$$\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}} . \quad \text{(Why?)}$$

In this form, $\frac{\sqrt{3}}{\sqrt{5}}$ involves two square roots of integers, and this certainly is not as simple as if it involved only one. How should we change $\frac{\sqrt{3}}{\sqrt{5}}$ so that there would be only one radical (with an integer under the radical sign) in the whole expression? We have two choices: We must somehow get rid of either the $\sqrt{3}$ or the $\sqrt{5}$. And how, for instance, might we get rid of $\sqrt{3}$? Let us recall the definition of $\sqrt{3}$. It is a number which when used as a factor twice gives 3. "...used as a factor..."; this is the clue. If we were to multiply the whole expression by $\frac{\sqrt{3}}{\sqrt{3}}$, which is just another way of saying 1, then in the numerator we would get $\sqrt{3}\sqrt{3}$, which is just 3, and the radical would be gone. In the denominator we would have $\sqrt{5}\sqrt{3}$, which is $\sqrt{15}$, and with this we can do no more, since 15 contains no perfect square factors. If we write this argument out in sequence (give the justification for each step), we have

$$\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}\sqrt{3}}{\sqrt{5}\sqrt{3}} = \frac{3}{\sqrt{15}}.$$

We have done what we set out to do; there is only one radical left. We did remark, however, that we had another choice. We could get rid of the $\sqrt{5}$ instead. In this case (justify each step),

$$\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{3}\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{\sqrt{15}}{5}.$$

Which of our two final expressions would we prefer? Each of them contains $\sqrt{15}$ and no other radicals, so that the argument we have used thus far will not help us to choose between them. People usually prefer the second of these, and here is why. If you had some sort of decimal approximation to $\sqrt{15}$, then the second form leaves you with an easy division to do, while the first one would lead to a difficult division. You will see how to find such an approximation later in this chapter, but suppose that you were told that $\sqrt{15}$ is approximately equal to 3.873.

How would you find a numerical approximation to $\sqrt{\frac{3}{5}}$? The form $\frac{3}{\sqrt{15}}$ would leave you with the problem of computing $\frac{3}{3.873}$, which is tedious; the form $\frac{\sqrt{15}}{5}$ leads to $\frac{3.873}{5}$, which is easy to do and yields 0.775.

For this reason, then, the form which leaves the denominator rational is often preferable to the form which leaves the numerator rational. Quite naturally, the process which leads to a rational denominator is called "rationalizing the denominator"; applied to the expression $\sqrt{\frac{3}{5}}$, rationalizing the denominator leads to $\frac{\sqrt{15}}{5}$, as you have seen.

Exercises 8 - 11b.

1. Rationalize the denominator:

Example: If x is positive,

$$\sqrt{\frac{3}{x}} = \frac{\sqrt{3}}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{3x}}{|x|} = \frac{\sqrt{3x}}{x}.$$ (Note: Here x must be positive in order that $\sqrt{\frac{3}{x}}$ be a real number.)

(a) $\sqrt{\frac{1}{2}}$

(b) $\sqrt{\frac{1}{3}}$

(c) $\sqrt{\frac{1}{4}}$

(d) $\sqrt{\frac{1}{5}}$

(e) $\sqrt{\frac{1}{6}}$

(f) $2\sqrt{\frac{2}{3}}$

(g) $3\sqrt{\frac{2}{5}}$

(h) $5\sqrt{\frac{3}{8}}$

(i) $\sqrt{\frac{5}{18}}$

(j) $\sqrt{\frac{25}{12}}$

(k) $\frac{1}{4}\sqrt{\frac{16}{3}}$

(l) $5\sqrt{\frac{4}{75}}$

(m) $\sqrt{\frac{7}{30}}$

(n) $\frac{1}{5}\sqrt{\frac{75}{12}}$

(o) $\frac{\sqrt{7}}{2}$

(p) $\sqrt{\frac{a^2}{11}}$

(q) $\sqrt{\frac{2x^2}{7}}$

(r) $\frac{1}{x}\sqrt{\frac{5x^2}{18}}$

(s) $\sqrt[3]{\frac{1}{2}}$

(t) $\sqrt[3]{\frac{1}{3}}$

2. Rationalize the denominator: (a, b, c, d are positive).

(a) $\sqrt{\frac{a}{2}}$

(b) $\sqrt{\frac{3b}{5}}$

(c) $\sqrt{\frac{2c}{a}}$

(d) $\sqrt{\frac{2a}{7b}}$

(e) $\sqrt{\frac{3c}{2d}}$

(f) $\sqrt{\frac{a}{2} + \frac{5}{b}}$

(g) $\sqrt{\frac{3}{b} + \frac{4}{b^2}}$

(h) $\sqrt{1 + \frac{1}{b}}$

(i) $\sqrt{\frac{a^2b}{12c}}$

(j) $\frac{1}{2}\sqrt{\frac{4a^2}{b}}$

(k) $\frac{1}{3}\sqrt{\frac{4c^2}{d^3}}$

(l) $\frac{1}{2ab}\sqrt{\frac{4a^2b}{b}}$

3. Rationalize the denominator:

   (a) $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{6}}$

   (b) $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{15}}$

   (c) $\dfrac{\sqrt{6}+\sqrt{24}}{\sqrt{6}}$

4. Simplify:

   (a) $(\sqrt{3}+\sqrt{2})^2$

   (b) $(3\sqrt{2}+2\sqrt{3})^2$

   (c) $(\sqrt{x}+1)^2$

   (d) $\left(\sqrt{a}+\dfrac{1}{\sqrt{a}}\right)^2$

Having seen what we would like to do with products and quotients of radicals in order to simplify them as much as possible, we turn next to sums and differences. We know that $\sqrt{2}$ and $\sqrt{3}$ each is irrational and that neither can be simplified any further. But is their sum rational? Try it; if $(\sqrt{2}+\sqrt{3})$ were a rational number, could you **square it**, (see problem 4 (a) above) and conclude something about $\sqrt{6}$? Is this conclusion about $\sqrt{6}$ true? What do you conclude about your assumption that $(\sqrt{2}+\sqrt{3})$ is rational? In the same way, you can see that no rational combination of $\sqrt{2}$ and $\sqrt{3}$ can be rational. If, therefore, you have reduced a simplification problem to the point where the expression reads $a\sqrt{2}+b\sqrt{3}+c$, where a, b, and c are rational expressions, then you have gone as far as you can go.

On the other hand, let us try, for example, $4\sqrt{3} - \frac{3}{2}\sqrt{12}$. Since 12 contains a square factor, you are not through. By the familiar procedure, you obtain:

$$4\sqrt{3} - \frac{3}{2}\sqrt{12} = 4\sqrt{3} - \frac{6\sqrt{3}}{2} = 4\sqrt{3} - 3\sqrt{3} = \sqrt{3}.$$

The last equality follows from the distributive law. It appears that we were able to simplify considerably in this case.

What is the key difference between these two examples? If you have a sum of different radicals, no one of which contains a perfect square factor, are you then finished? If one or more radicals does contain a perfect square factor, can you be sure that nothing more can be done?

Exercises 8 - 11c.

1. Simplify:

(a) $\sqrt{2} + \sqrt{8}$

(b) $\sqrt{18} - \sqrt{27}$

(c) $2\sqrt{12} + \sqrt{8}$

(d) $\sqrt{27} - \sqrt{18}$

(e) $8\sqrt{\frac{1}{2}} - \sqrt{32}$

(f) $\sqrt{16} + \sqrt{48}$

(g) $\frac{2}{3} + \frac{1}{\sqrt{6}}$

(h) $\sqrt{50} - \sqrt{98} + \sqrt{8}$

(i) $\sqrt{\frac{1}{3}} - 3\sqrt{3} + \sqrt{27}$

(j) $\sqrt{24} + \sqrt{54} + \sqrt{63}$

(k) $\sqrt{5x^2} - \sqrt{50x^2} + |x|\sqrt{45}$

(l) $\sqrt{34} + \frac{1}{2}\sqrt{16} - \sqrt{20}$

(m) $\frac{1}{4}\sqrt{288} - \frac{1}{6}\sqrt{72} + \sqrt{\frac{1}{24}}$

(n) $\sqrt{61} - \sqrt{73}$

(o) $\sqrt{\frac{1}{7}} + \frac{\sqrt{5}}{\sqrt{17}}$

(p) $\sqrt{\frac{3}{28}} - 3\sqrt{24} + \sqrt{\frac{23}{16}}$

(q) $\sqrt{150} + \sqrt{\frac{8}{3}} - 2\sqrt{54}$

(r) $\frac{\sqrt{40}}{18} - \frac{\sqrt{90}}{12} - \frac{\sqrt{160}}{6}$

(s) $\frac{\sqrt{56}}{18} + \frac{\sqrt{2}}{2\sqrt{7}} - \frac{\sqrt{28}}{12\sqrt{2}}$

(t) $\sqrt{3}\sqrt{6} + \sqrt{5}\sqrt{10}$

(u) $\sqrt{2}\sqrt{12} - \frac{\sqrt{2}}{\sqrt{3}}$

(v) $\frac{\sqrt{2}\sqrt{3}}{\sqrt{5}} + \frac{\sqrt{60}}{\sqrt{2}}$

2. Simplify, for positive a, b, c:

(a) $\sqrt{a} + 2\sqrt{a}$

(b) $\sqrt{9a} + \sqrt{4a}$

(c) $\sqrt{a}\sqrt{b} - \sqrt{4ab}$

(d) $\frac{\sqrt{a}}{\sqrt{b}} + \sqrt{ab}$

(e) $\sqrt{ab^2c^3} - \sqrt{a}\sqrt{c}$

(f) $\sqrt{12a} - \frac{1}{3}\sqrt{48a}$

3. For what values of x are the following sentences true?

(a) $x^2 = 5$

(b) $x^2 = \frac{1}{2}$

(c) $x^2 = 24$

(d) $2x - \sqrt{18} = x + \sqrt{32}$

(e) $x\sqrt{2} = \sqrt{3}$

(f) $\sqrt{x} = 5$

(g) $\sqrt{x^2} = 3$

8 - 12. *Square roots*. You have learned to recognize instantly the square roots of 1, 4, 9, 16, 36, 49, 64, 81, 100, 121 and 144. You may even be able to identify the square roots

of 225, 400, 625, 900, etc. But what is the square root of 31200? Of .0621? Of .0000123? We shall try to answer these questions in this section.

We know that the square root of 81 is a rational number, because 81 is a perfect square. We also know that the square root of a number which is not a perfect square, such as 29, is irrational, that is, $\sqrt{29}$ cannot be written as the quotient of two integers. What we mean by "evaluating $\sqrt{29}$" is finding a rational number (in the form of a decimal) as close to $\sqrt{29}$ as is required. A rational number close to $\sqrt{29}$ is called an approximation of $\sqrt{29}$. The difference between $\sqrt{29}$ and its rational approximation is called the error of the approximation.

There are two stages in the process of evaluating a square root. First, we must make an "educated guess" or estimate of a first approximation. Second, we must refine our first approximation by a series of steps designed to bring our approximations as close to the value of the square root as we desire.

In order to make a rough estimate of the value of the square root of a number, we should first put the number in a standard form. By this we mean write the number as the product of a number M between 1 and 100 and a power of 10. For example,

$$392 = 39.2 \times 10 = 3.92 \times 10^2$$
$$3920 = 39.2 \times 10^2 = 3.92 \times 10^3$$
$$39200 = 39.2 \times 10^3 = 3.92 \times 10^4$$

Notice that each of these numbers can be written either as 39.2

times a power of ten or as 3.92 times a power of 10.

If we define 10^0 to be 1, 10^{-1} to be 1/10, 10^{-2} to be $1/10^2$, etc., then we may write

$$0.392 = 39.2 \times 10^{-2} = 3.92 \times 10^{-1}$$

$$0.0392 = 39.2 \times 10^{-3} = 3.92 \times 10^{-2}$$

$$0.00392 = 39.2 \times 10^{-4} = 3.92 \times 10^{-3}$$

You see that any number can be written in two forms as

$$M \times 10^k,$$

where M is a number between 1 and 100 and k is an integer. In one of the forms, k is an even integer; in the other, k is odd. Write 29300 in two standard forms. Do the same for 0.000293. For 0.00293.

Now we are ready to make a rough estimate of the square root of a number. It will depend on our understanding that $\sqrt{10^2} = 10$, $\sqrt{10^4} = 10^2$, $10^6 = 10^3$, etc., and $10^{-2} = 10^{-1}$, $10^{-4} = 10^{-2}$, etc. What is 10^0 ? Make up a general rule for the square root of an even power of 10. Let us make an estimate of 354000. We know that

$$354000 = 35.4 \times 10^4 = 35.4 \quad 10^4 = 35.4 \times 10^2.$$

(Why did we choose $354000 = 35.4 \times 10^4$ rather than 3.54×10^5 ?) Now our problem is narrowed down to an estimate of the square root of a number between 1 and 100. What is your estimate of 35.4 ? Certainly you will say "6", because $6^2 = 36$, and 36 is close to 35.4. We will always be able to estimate to the nearest integer the value of the square root of a number between 1 and 100. So our estimate of 354000 is 6×10^2 or 600. Next, let us estimate 0.00000853. Since

$$0.00000853 = 8.53 \times 10^{-6} = 8.53 \times 10^{-3},$$

we estimate 8.53 as 3, because $3^2 = 9$. Then an estimate of 0.00000853 is $3 \times 10^{-3} = 0.003$.

Exercises 8 - 12a.

1. Write each of the following in a standard form involving an even power of 10:

   (a) 33
   (b) 0.0726
   (c) 5280
   (d) 3.1416
   (e) 0.00823
   (f) 70260
   (g) 0.273
   (h) 0.000470
   (i) 502060

2. Estimate the following:

   (a) $\sqrt{796}$
   (b) $\sqrt{73}$
   (c) $\sqrt{0.0604}$
   (d) $\sqrt{304007}$
   (e) $\sqrt{0.00580}$
   (f) $\sqrt{9999900}$
   (g) $\sqrt{0.507}$
   (h) $\sqrt{1003}$
   (i) $\sqrt{0.0000001}$

3. (Optional) To estimate the cube root of a number, write the number in a standard form involving a power of 10 which is a multiple of 3, etc. What is an estimate of $\sqrt[3]{0.0260}$? Of $\sqrt[3]{7806}$?

After we have estimated the square root, that is, obtained a first approximation of the square root, we need a technique for improving the approximation.

If we want to approximate the value of n, and if x = n, then $x^2 = n$; i.e., $x \cdot x = n$. (Why?) We have already made an estimate of x, say x_1. Then

$$x_1 \cdot z = n$$

If the estimate x_1 is too large, then the other number z must be too small. If x_1 is too small, then z is too large. It follows, therefore, that $\sqrt{n}$ lies between z and x_1. Since $z = \frac{n}{x_1}$, we know that $\sqrt{n}$ lies between $\frac{n}{x_1}$ and x_1. Thus a likely next approximation of $\sqrt{n}$ is half way between $\frac{n}{x_1}$ and x_1. So we take as our next approximation, their average, $\frac{1}{2}\left(\frac{n}{x_1} + x_1\right)$.

Let us try this technique for $\sqrt{31200}$. Since

$$\sqrt{31200} = \sqrt{3.12 \times 10^4} = \sqrt{3.12} \times 10^2,$$

we are concerned with the value of $\sqrt{3.12}$. A first estimate of $\sqrt{3.12}$ is 2. (Why?) Thus, $\sqrt{3.12}$ lies between $\frac{3.12}{2}$ and 2, i.e., between 1.56 and 2. Their average is $\frac{1}{2}(1.56 + 2) = 1.78$. This means that $\sqrt{3.12} \approx 1.78$ (the symbol $\approx$ means "is approximately equal to"), or $\sqrt{31200} = \sqrt{3.12} \times 10^2 \approx 1.78 \times 10^2 = 178$.

It can be shown that this second approximation may be in error in the third digit. If we want an even closer approximation, we may again "divide and average". We could use our second approximation, 178, as the divisor; but for practical purposes it is sufficient to use a two digit divisor. Thus $\sqrt{3.12}$ lies between $\frac{3.12}{1.8}$ and 1.8, i.e., between 1.733 and 1.8. Averaging these, a third approximation is

$$\frac{1}{2}(1.733 + 1.8) = 1.767.$$

Thus, $\sqrt{31200} \approx 176.7$. Following the procedure outlined above will, in general, give an approximation which is in error by no more than 3 in the fourth digit. For this reason, we stopped the division of 3.12 by 1.8 when we had obtained four digits, 1.733.

This process of evaluating a square root can be put into table form. As another example, let us evaluate $\sqrt{0.0029}$.

$$\sqrt{0.0029} = \sqrt{29 \times 10^{-4}} = \sqrt{29} \times 10^{-2}.$$

Since $5^2 = 25$ and $6^2 = 36$, the closest integer is 5; so we take 5 as the first estimate.

Approx. x	$\frac{29}{x}$	$\frac{1}{2}\left(\frac{29}{x} + x\right)$
5	5.80	5.40
5.4	5.370	5.385

A table of square roots gives, to 7 digits: $\sqrt{29} \approx 5.385165$; this shows that our second approximation, 5.40, is in error by .015, and our third approximation is correct to four digits. Thus,

$$\sqrt{0.0029} \approx 0.05385.$$

We may observe several facts from the above table, all of which can be proved: (1) each new approximation is greater than the desired root, and (2) each new approximation has about twice the number of correct digits as the previous approximation.

Summary: To evaluate $\sqrt{n}$, first write it as the square root of a number between 1 and 100 times an even integer power of ten: $\sqrt{n} = \sqrt{M} \times 10^{2p} = \sqrt{M} \times 10^{p}$, $1 < M < 100$, p an integer.

Second, estimate $\sqrt{M}$ to the nearest integer x_1. Then a second approximation x_2 is obtained by dividing and averaging:

$$x_2 = \frac{1}{2}\left(\frac{M}{x_1} + x_1\right).$$

Carry out the division, $\frac{m}{x_1}$ to three digits and remember that the third digit of x_2 may be in error. If more accuracy is wanted, round x_2 to two digits and repeat the process of dividing and averaging. x_3 will be in error by no more than 3 in the 4th digit; usually the error is no more than 1 in the 4th digit. If still more accuracy is desired, repeat the process of dividing and averaging (but do not round off the divisor). The number of correct digits will double each time.

Exercises 8 - 12b.

1. Use the estimates obtained in problem 2 of Exercises 8 - 12a to find the second approximations of:

   (a) $\sqrt{796}$
   (b) $\sqrt{73}$
   (c) $\sqrt{0.0684}$
   (d) $\sqrt{304007}$
   (e) $\sqrt{0.00580}$
   (f) $\sqrt{9999900}$

2. Find the third approximations of:

   (a) $\sqrt{0.00470}$
   (b) $\sqrt{0.273}$
   (c) $\sqrt{5280}$
   (d) $\sqrt{3.1416}$
   (e) $\sqrt{70260}$
   (f) $\sqrt{502060}$

3. If you had a table listing approximations of the square roots of all the integers from 1 to 100, explain how you would use this table to find an approximation of $\sqrt{0.0072}$? Of $\sqrt{280,000}$? Of $\sqrt{600}$?

4. Find the truth set of each of the following: (to the second approximation)

(a) $x^2 = 0.0124$ (b) $x^2 - 519 = 0$

5. A fourteen foot ladder rests against a vertical wall, the foot of the ladder being seven feet from the base of the wall. Determine the height at which the ladder touches the wall. (Find the third approximation of the result.)

(Optional) The procedure we have been following for approximating a square root seems to work. It seems to give rational numbers which are closer and closer to the irrational square root. The question is: How close? Let us reason as follows. If x_1 is a positive approximation to $\sqrt{n}$ such that $x_1 > \sqrt{n}$, then

$$x_1^2 > n, \qquad \text{(Why?)}$$

and

$$x_1 > \frac{n}{x_1}. \qquad \text{(Why?)}$$

Then, by adding x_1 to both sides,

$$2x_1 > x_1 + \frac{n}{x_1}$$

and

$$x_1 > \frac{1}{2}\left(x_1 + \frac{n}{x_1}\right). \text{(Why?)}$$

Since $x_2 = \frac{1}{2}\left(x_1 + \frac{n}{x_1}\right)$, we have shown that the second approximation is always less than the first.

Let the difference between an approximation and $\sqrt{n}$ be called the error e of the approximation. Then

$$e_1 = x_1 - \sqrt{n}$$

and

$$e_2 = x_2 - \sqrt{n}.$$

Thus,

$$e_2 = \frac{1}{2}\left(x_1 + \frac{n}{x_1}\right) - \sqrt{n}.$$

By adding fractions on the right and commuting terms,

$$e_2 = \frac{x_1{}^2 - 2\sqrt{n}\,x_1 + n}{2x_1}.$$ (Fill in the steps.)

But the numerator on the right is a perfect square:

$$e_2 = \frac{(x_1 - \sqrt{n})^2}{2x_1}.$$

Now we observe these facts:

(1) The error e_2 in the second approximation is positive, because the square of any non-zero number is positive. Hence, x_2 is greater than $\sqrt{n}$. Why? Then $\sqrt{n} < x_2 < x_1$, and we have shown that x_2 is closer to $\sqrt{n}$ than is x_1.

(2) The same procedure would give us the error of any approximation x in terms of the preceding approximation z,

$$\text{error of } x = \frac{(z - \sqrt{n})^2}{2z},$$

and this error is always positive. Then x is closer to $\sqrt{n}$ than is z, and we may replace $\sqrt{n}$ by x and get the approximate formula for the error:

$$\text{error of } x \approx \frac{(z - x)^2}{2z} ,$$

where z is the larger of the previous approximation or n divided by the previous approximation.

Going back to our approximations of $\sqrt{29}$, note that $x_3 = 5.385$ and $x_2 = 5.4$; hence the error x_3 is

$$e_3 \approx \frac{(5.4 - 5.385)^2}{2(5.4)} ,$$

$$e_3 \approx \frac{(.015)^2}{10.8} \approx 0.00002.$$

This means that x_3 is larger than $\sqrt{29}$ by about 0.00002. It shows that we could have computed x_3 to more digits. If we compute x_3 to six digits, we have

$$x_3 = \frac{1}{2}\left(5.4 + \frac{29}{5.4}\right) = \frac{1}{2}(5.4 + 5.37037) = 5.38518.$$

This is too large by about 0.00002; by subtracting the error we have

$$\sqrt{29} \approx 5.38516.$$

If the error in each approximation is taken into account, one can obtain a large number of correct digits very quickly. Let us evaluate $\sqrt{31200}$ again as an example. Since $\sqrt{31200} = \sqrt{3.12} \times 10^2$, we compute $\sqrt{3.12}$.

Approx. x	$\frac{3.12}{x}$	$\frac{1}{2}\left(\frac{3.12}{x} + x\right)$	Approx. Error of x.
2	1.56	1.78	$\frac{(.22)^2}{2(2)} = .01$
1.78 - .01 = 1.77	1.762711	1.766355	$\frac{(0.004)^2}{3.6} = .000004$

$$\sqrt{31200} \approx 17.66355 - .000004 = 17.66351.$$

You might wonder how far to carry out the division $\frac{3.12}{1.77}$. The error of x is given by

$$e_3 \approx \frac{(z - x)^2}{2z} .$$

$$z - x = z - \frac{1}{2}\left(\frac{n}{z} + z\right)$$

$$= z - \frac{n}{2z} - \frac{z}{2}$$

$$= \frac{z}{2} - \frac{n}{2z} .$$

$$z - x = \frac{1}{2}\left(z - \frac{n}{z}\right).$$

In this example $z = 1.77$ and $\frac{n}{z} = \frac{3.12}{1.77} = 1.762 \cdots$. When $\frac{n}{z}$ is carried out far enough so its digits begin to differ from the digits of z, we can find $z - x$. In this case

$$z - x = \frac{1}{2}\left(z - \frac{n}{z}\right) = \frac{1}{2}\left(1.77 - 1.762\right) = .004.$$

Now we can find e_3 approximately;

$$e_3 \approx \frac{(.004)^2}{2(1.8)} = .000004.$$

Now we know that if we continue the division, $\frac{n}{z}$, to six decimal places and average, the error will occur in the sixth decimal place.

Exercises 8 - 12c.

1. Find a second approximation to each of the following, correcting each approximation:

(a) $\sqrt{8.3}$ (c) $\sqrt{.17}$ (e) $\sqrt{32.41}$

(b) $\sqrt{.51}$ (d) $\sqrt{169}$ (f) $\sqrt{.053}$

2. Find the truth set of each of the following: (determine solutions correct to 4 digits)

(a) $x^2 = 386$ (c) $x^2 = 0.0124$

(b) $x^2 - 519 = 0$ (d) $x^2 - 0.792 = 0$

3. If, in the calculation of a square root, an approximation z of the root is 52, and if the error, e, in z is known to be between .2 and .3, then between what values does $\frac{e^2}{2z}$ lie?

4. (a) Approximate the square root of 2680 to the nearest ten's.

(b) Using this approximation, find a second, closer approximation to the square root of 2680 (written to three digits only.)

(c) Write a third approximation to at least six digits.

(d) To how many digits is this third approximation correct?

5. Show that for positive n and x

$$\frac{1}{2}\left(x + \frac{n}{x}\right) \geq \sqrt{n}.$$

(Hint: Note that the error

$$e = \frac{1}{2}\left(x + \frac{n}{x}\right) - \sqrt{n}$$

is always positive or zero.) For what value of x does the equality hold?

8 - 13. Review Exercises.

1. Express as the product of prime numbers:

(a) 24	(c) 360	(e) 1225
(b) 363	(d) 2844	(f) 2310.

2. Find the least common multiple of each of the following sets of numbers:

(a) 24, 36	(c) 8, 24, 40
(b) 24, 108	(d) 75, 45, 500

3. Find the greatest common factor of each of the sets of numbers in problem 2.

4. A famous problem states that every even number greater than 2 is the sum of a pair of prime numbers. For example $4 = 2 + 2$, $6 = 3 + 3$, $8 = 5 + 3$, $18 = 13 + 5$, etc. Can you find an even number less than 100 for which the statement is not true? (Goldbach in 1742 made this original statement which has never been proved or disproved.)

5. The primes which differ by 2 are 3 and 5, 11 and 13, 29 and 31, etc. These are called prime pairs. List all the prime pairs among the numbers less than 100. (It is still not known whether the set of prime pairs is finite.)

6. A remarkable expression which produces many primes is

$$n^2 - n + 41.$$

If n is any number of the set $\{1, 2, 3, \cdots, 40\}$ the value of the expression is a prime number, but for $n = 41$ the expression fails to give a prime number. Tell why it fails. If an algebraic sentence is true for the first 400 values of the variable, is it then necessarily true for the 401st?

In problems 7 through 28 simplify when possible:

7. $\frac{5}{12} + \frac{7}{36} - \frac{13}{54}$

8. $\sqrt{3a}\sqrt{6a^3}$

9. $(3^3)(9^2) + 2(3^2) - (4)(2^3) - (3^3)(9^2) + (2 + 3)^2$

10. $\sqrt{48} - \sqrt{75} + \sqrt{12}$

11. $\left(\frac{21a^3b^2}{4cd^2}\right)\left(\frac{12c^4d^3}{15a^4b}\right)$

12. $(3^a)(2^b)$

13. $\sqrt{\frac{1}{2}} - \frac{\sqrt{9}}{\sqrt{8}}$

14. $\sqrt{4(a + b)^2}$

15. $\frac{7}{18x^2} - \frac{9}{30x^2y^2} + \frac{13}{20y^2}$

16. $\dfrac{\dfrac{30x^3y^4}{28ab^2}}{\dfrac{14ay^3}{3bx}}$

17. $\sqrt{4a^2 + 4b^2}$

18. $(32)(2x^4) + (-3x)^3(2x) + 2y^6 + (-2y^2)^3$

19. $\dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{2}}$

20. $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$

21. $\dfrac{\dfrac{ab^2}{x^2y^2}}{\left(\dfrac{ab}{xy}\right)^2}$

22. $-2\sqrt{3}(2 - \sqrt{6})$

23. $\sqrt{|(\sqrt{16} - \sqrt{25})|}$

24. $\dfrac{\sqrt{3}}{\sqrt{2}} + 1$

25. $-\sqrt{32x^3y}$

26. $(\sqrt{\frac{2}{3}})(\sqrt{\frac{5}{6}})(\sqrt{\frac{3}{2}})$

27. $\sqrt[3]{-8} + (-\sqrt[3]{8}) + \sqrt[3]{64}$

28. $\sqrt{(7)^2 + 2(7)(5) + (5)^2}$

In problems 29 through 35, find the truth sets of the sentences given:

29. $\sqrt{x} = 2$

30. $m^2 \leq m$

31. $\sqrt[3]{n} = 4$

32. $x^2 = \frac{1}{2}$

33. $3 = \sqrt{t + 1}$

34. $2|x| + \sqrt{x^2} = 3$

35. $-|x^2 + 2| > \sqrt{x^2 + 2}$

In each of the problems 36 through 42, use one of the symbols $<$, $=$, $>$ between the two given phrases so as to make a true sentence.

36. $\frac{1}{x} + \frac{2}{3x}$, $\frac{1}{3}$, for $x = 5$

37. $x + \sqrt{2}$, $\sqrt{2}$, for $x > 0$

38. $\sqrt{a^2 + b^2}$, $(a + b)$, for $a > 0$, $b > 0$

39. $\sqrt[4]{3^2}$, $\sqrt{3}$,

40. $(\sqrt{m} + \sqrt{n})(\sqrt{m} - \sqrt{n})$, $(m-n)$, for $m > 0$, $n > 0$

41. $(|x| + 5\sqrt[3]{125})$, $(-x)^3$, for $x = 25$

42. $\frac{1}{2} - \frac{\sqrt{9}}{\sqrt{8}}$, $(2\sqrt{8} - \sqrt{50})$,

43. Find the prime factors of 442.

44. Is the set of numbers of the form $n + m\sqrt{2}$, where n and m are integers, closed under addition? Under multiplication?

45. Is the set of numbers of the form $n + m\sqrt{2}$, where n and m are rational numbers, closed under addition? Under multiplication?

46. Is it possible to find rational numbers m and n in problem 45 so that $n + m\sqrt{2}$ is a rational number? Prove your statement.

47. If, in problem 46, $m \neq 0$, are all the numbers $n + m\sqrt{2}$ irrational? Is the set finite or infinite? Does it follow from this that there are infinitely many irrational numbers?

48. Evaluate $\sqrt{390}$ (to the second approximation).

49. Evaluate $\sqrt{3900}$ (to the third approximation).

50. Evaluate $\frac{1}{\sqrt{70}}$.

51. (Optional) In evaluating $\sqrt{321}$, calculate the second approximation to six digits. What is the estimated error of this six-digit second approximation? Is this about the same as the actual error of this second approximation, being given that the value of $\sqrt{321}$ is $17.916472867\cdots$? (Note: Do not attempt to calculate this last value, although it can be obtained by continuing the process one step further.)

Chapter 9

Polynomials and Rational Expressions

9 - 1. **Monomial factors.** In the last chapter, we found that there were many advantages to having a number in factored form. When we consider the number 288, for example, we now have, besides the form $2(100) + 8(10) + 8$, also the form $2^5 \cdot 3^2$. If you want to tell whether or not 288 is a perfect square, which form would you use? What if you wanted to know the prime factors of 288? What if you wanted to find the simplest form for $\sqrt{288}$? in arithmetic, the form $2(100) + 8(10) + 8$ was good enough for most of our work; in algebra, the factored form $2^5 \cdot 3^2$ is often handier.

Since the prime factored form of integers has turned out to be so useful, we are tempted to look for other situations in which we can write expressions in "factored" form, that is, as products of simpler expressions. What property of the real numbers enables us to write, for any real number x,

$$x^2 + 3x = x(x + 3) \text{ ?}$$

Is each of the expressions x and $(x + 3)$ simpler, in some sense, than $x^2 + 3x$?

The operations we need to form $x^2 + 3x$ are addition and multiplication; the operations we need to form x and $(x + 3)$ are addition and multiplication; but when we try other factors whose product is $x^2 + 3x$, $7x(\frac{x}{7} + \frac{3}{7})$, or $(x^2 + 1)(\frac{x^2 + 3x}{x^2 + 1})$, they also involved division. We will not be satisfied with factors unless they involve only addition and multiplication, because this is all that was required in the original $x^2 + 3x$. And so, we give a name to expressions which we can obtain by performing additions and multiplications on real numbers, some of which may be represented by variables. We shall call such expressions polynomials. If there is just one variable, say "x", we shall say that we have a polynomial in one variable x, or just a polynomial in x. And if the numbers (other than those expressed as variables) we use in the formation of the polynomial are only integers, we shall say that we have a polynomial over the integers.

Now we see what attracted us to the form $x(x + 3)$ for $x^2 + 3x$. The expression $x^2 + 3x$ is a polynomial in x over the integers, and when we wrote it as $x(x + 3)$, we managed to write it as a product of polynomials over the integers, and this is what we meant by the simplicity of our factored form. Does either the product $7x(\frac{x}{7} + \frac{3}{7})$, or the product $(x^2 + 1)(\frac{x^2 + 3x}{x^2 + 1})$, use only polynomials over the integers?

Exercises 9 - 1a. In the light of the preceding discussion, write each of the following polynomials in factored form. State the variables in each polynomial, and a set of numbers over which the polynomial and its factors are defined.

1. $3y^2 - 6y$. This is a polynomial in y over the integers and can be written as $3y(y - 2)$, where both $3y$ and $y - 2$ are the same kind of polynomial.
2. $\frac{3}{2}st + \frac{15}{2}st^2 - 27s^2t^3$. This is a polynomial in two variables s and t over the rational numbers. This can be written in

various forms, of which some are: (by what properties of real numbers?)

(a) $\frac{1}{2}(3st + 15st^2 - 54s^2t^3)$

(b) $3s(\frac{1}{2}t + \frac{5}{2}t^2 - 9st^3)$

(c) $st(\frac{3}{2} + \frac{15}{2}t - 27st^2)$

(d) $\frac{3}{2}st(1 + 5t - 18st^2)$.

Which of these is simplest? This is just like the question we asked about various factorizations of numbers: The simplest form was that in which no more factoring was left to do. In (a), the distributive property still permits us to "factor out" a "3" and an "s" (and what else?); in (b) there is still a "t", in (c) there is still a "3" and there are still some unnecessary fractions. In (d), we have done all we can as far as we now know; we do not know whether $1 + 5t - 18st^2$ can be simplified any further, but it certainly cannot be factored by any additional application of the distributive property. Notice that in working with a polynomial over the rational numbers, we obtain the simplest expression if we write it as a product of polynomials over the integers times one (non-integral, if necessary) rational number.

3. $a^2 + 2ab$

4. $\frac{7}{8}x^3 - \frac{35}{4}x^2y + \frac{21}{2}xy^2$

5. $6x^2 - 144x - 150$

6. $3x(2xz - 4yz)$

7. $16ax^2 - 27$

8. $144x^2 - 216s + 180y$. What have you learned to do with integers that will enable you to find the largest common factor here?
9. $\frac{6}{5}a^2b - \frac{9}{5}ab^2 + 3b^2$
10. $-x^3y^2 + 2x^2y^2 + xy^2$
11. $36a^2b^2c^2$
12. $\frac{1}{6}ab + \frac{5}{18}a^2b - \frac{7}{12}ab^2$
13. $x^2\sqrt{2} - 3x\sqrt{2}$
14. $s\sqrt{3} + s^2\sqrt{6}$. What have you learned about $\sqrt{6}$ which you can use here?
15. $12xy\sqrt{\frac{3}{2}} - 5x^2y^2\sqrt{6} + 6xy^2\sqrt{24}$.

We can now see what "factoring" is going to mean: if we have a polynomial over the integers, we want to write it as a product of polynomials over the integers, and we are not happy until we can no longer do this to any of the polynomials which appear in our product. (Remember that in the case of positive integers, we were not happy until we had all prime factors.) Each of the resulting polynomials is called a "factor" of the original polynomial.

If the polynomial is over the rational numbers, we obtain the simplest factors by writing it as a product of polynomials over the integers and one rational number. If irrational numbers are involved in our polynomial, we may not be able to produce factors which are polynomials over the integers, but we come as close to this as we can, according to our knowledge of radicals.

Exercises 9 - 1b.

Simplify:

1. $a\sqrt{3} - ab\sqrt{12}$
2. $6xy + 2y(x - 3)$
3. $\frac{a}{2}\sqrt{\frac{1}{2}} - \frac{2a}{3\sqrt{2}}$
4. $3ab + 4bc - 5ac$
5. $54a - 24ab + 180ac$
6. $6a\sqrt[3]{2} + 15ab\sqrt[3]{2}$
7. $3|x| + 2a|x|$
8. $7y\sqrt{x^2} - 21y^2|x|$
9. $a(x - 1) + 3(x - 1)$. Can $(x - 1)$ be considered a single number?

We have now found one situation in which it is possible to factor a polynomial, namely the case in which the distributive law lets us see immediately what to do, because there is a common factor throughout the polynomial on which we were working. In problem 1 of Exercises 9-1a, $3y$ was such a factor; in problem 2, it was $\frac{3}{2}st$. What was this common factor in problems 3 and 4? Have you noticed something interesting about each of these common factors? Each of them is a polynomial, but it is formed by multiplication only (think back to our definition of polynomial), and so it is a very special kind of polynomial, which is called a *monomial*. Thus a catch phrase for the kind of factoring we have just done is *common monomial factoring*.

We should, while we have the opportunity, discuss one other aspect of monomials. A monomial is a product of a real number and variables; if we single out any factor of this product, what remains is called the coefficient of that factor. In $\frac{3}{2}st$, for example, $\frac{3}{2}s$ is the coefficient of t, $\frac{3}{2}t$ is the coefficient of s, and $\frac{3}{2}$ the coefficient of st.

The distributive property under the fancier title "common monomial factoring", has enabled us to factor polynomials like $x^2 + bx$ and $ax + ab$ into

$$x^2 + bx = x(x + b)$$

and

$$ax + ab = a(x + b),$$

respectively. Suppose now that we were interested in the polynomial

$$x^2 + bx + ax + ab.$$

We have agreed to call expressions joined by a "+" symbol terms. Now you see that you can factor the sum of the first two terms, namely $x^2 + bx$, and the sum of the last two terms, $ax + ab$. Can you factor the sum of all four terms? Do you remember a theorem in Chapter 8 about factors of a sum? What is a factor of both $x^2 + bx$ and $ax + ab$? What is the sum of $x^2 + bx$ and $ax + ab$?

Thus $(x + b)$ must be a factor of

$$x^2 + bx + ax + ab,$$

and, by the distributive property applied three times,

$$\begin{aligned} x^2 + bx + ax + ab &= x(x + b) + a(x + b) \\ &= (x + a)(x + b). \end{aligned}$$

Exercises 9 - 1c.

Using the properties of the real numbers, write the following polynomials in factored form. (We will frequently just say "factor").

1. $2st + 6xy - 3sx - 4yt$. If we begin as before, we can write the sum of the first two terms as $2(st + 3xy)$, but then we can go no further because the sum of the last two terms isn't factorable. (What do we mean, precisely, by a polynomial not being factorable?) The reason we haven't got very far is that the first two terms, taken together, do not have very much in common; such a situation does not lend itself well to common monomial factoring. The commutative property of addition, however, permits us to reorder the terms if we so choose. We can, for example, write
$$2st + 6xy - 3sx - 4yt = 2st - 3sx + 6xy - 4yt.$$
Can you now finish the problem? Try it again, but now interchange the second and fourth terms before you begin applying the distributive law. Are the two factored forms the same?
2. $x - 4yz - 4y + xz$
3. $a^2 - 4ax + 2ab + 3ac - 12cx - 8bx$
4. $5x + 3xy - 3y - 5$
5. $3x - 6x^2 + 45xy^2 - 12xz$
6. $2a^2 - 2ab\sqrt{3} - 3ab - 3b^2\sqrt{3}$
7. $15ax + 12bx - 9cx + 6dx$
8. $3a + 15b - 3a - 15b$
9. $a^2 - ab + ac + bc$

10. $\frac{1}{2}axy - a^2y + \frac{1}{6}abx - \frac{1}{3}a^2b$

11. $3x - 6x^2 + 45xy^2 - 12xz$

12. $s^2 - 4s + 3s - 12$

9 - 2. Polynomial factors. Notice that the terms are not collected in the polynomial of problem 12 in the previous set of exercises. Because of the way it is written we have no trouble factoring $s^2 - 4s + 3s - 12$; but suppose that we had combined the middle terms (by what property of real numbers) and written $s^2 - s - 12$? Can we now factor

$$s^2 - s - 12?$$

Of course we know that $s^2 - s - 12$ can be factored! We can write it as

$$s^2 - 4s + 3s - 12,$$

and this can be factored as before. But, if we had not come from this second form to begin with, how would we guess that the "$-s$" in the middle ought to split into

$$-4s + 3s?$$

After all, $-3s + 2s$ or $7s + 6s$ would not have worked!

It will take us some time to learn to answer this kind of question. Let us begin by getting acquainted with some terminology which we shall need. We can think of the polynomial

$$s^2 - s - 12$$

also in the form

$$s^2 + (-1)s + (-12).$$

We see that it consists of three terms: "s^2" is called the quadratic term, "$(-1)s$" the linear term, and "-12" the constant term of the polynomial. Such a polynomial is called quadratic because it involves a quadratic term. Similarly, if we consider the polynomial

$$2x^2 + 11x + 15,$$

then "$2x^2$" is the quadratic term, "$11x$" the linear term, and "15" the constant term. In the linear term, 11 is the coefficient of x, while in the quadratic term, 2 is the coefficient of x^2. What are the coefficients of the linear and quadratic terms of $s^2 - s - 12$?

By using the distributive law several times, we can take the factored form $(s - 4)(s + 3)$ and write

$$\begin{aligned}(s - 4)(s + 3) &= s^2 - 4s + 3s - 12\\ &= s^2 - s - 12.\end{aligned}$$

How did the quadratic term "s^2" arise? As the product of s and s. How did the constant term "-12" arise? As the product of -4 and 3. How did the linear term "$-s$" arise? As the sum of $-4s$ and $3s$.

Can you predict the distributed form of

$$(y + 6)(y + 8)?$$

Can you say what the product $(x + a)(x + b)$ will yield?

Our real interest now is to proceed in the other direction, that is, to factor rather than to form products. Let us use what we have learned about the formation of coefficients, and factor

$$t^2 + 5t + 6$$

We want to write this in the form $(t + a)(t + b)$, where a and b are numbers. What properties will a and b have to have? Their product, ab, must equal 6, while their sum, $a + b$, must equal 5. Can you find two numbers whose product is 6 and whose sum is 5?

This is easy: 6 can be factored into 6 and 1, or 3 and 2; but $6 + 1 \neq 5$, while $3 + 2 = 5$. So we know that

$$t^2 + 5t + 6 = (t + 3)(t + 2).$$

Exercises 9 -2a.

Factor:

1. $x^2 + 22x + 72$. This means we want two factors of the number 72 whose sum is 22. Now 72 has many factors, and we would prefer to solve our problem without having to try them all. Can you think of any properties of factors to help you out? Doesn't this specific problem sound vaguely familiar? We answered this precise question in Chapter 8, in the course of our study of prime factorization of integers. Go back and look at the discussion again. We found that $72 = 2^3 \cdot 3^2$, and that in the two factors of 72 which we were seeking, the 2's had to be split, two in one factor and one in the other, while the 3's had to be together. The factors we found by this argument were 18 and 4. If we apply this in our present situation, this tells us immediately that

$$x^2 + 22x + 72 = (x + 18)(x + 4).$$

2. $y^2 - 14y + 45$. In this problem, we require two numbers whose product is 45, and whose sum is -14. Can two positive

numbers have the sum -14? Can one positive and one negative number have the positive product 45? However, two negative numbers have a positive product, and so we shall be able to write $y^2 - 14y + 45$ in the form $(y + p)(y + q)$ if we find two negative numbers p and q whose product is 45, and whose sum is -14. This means that the opposites of p and q have the sum 14, and have a product equal to 45 (why?) Can we now find these opposites of p and q as before? The prime factorization of 45 is $3^2 \cdot 5$. 14 is not divisible by 3; hence, we cannot split the two 3's. (Why?) Thus the factors are either 9 and 5, or 45 and 1; clearly, 9 and 5 are the required numbers. Thus $-p$ and $-q$ are 5 and 9 respectively. Then

$$y^2 - 14y + 45 = (y - 5)(y - 9).$$

3. $x^2 + 10x + 12$. We need two factors whose product is 12 and whose sum is 10. The prime factorization of 12 contains two 2's and one 3; the two 2's have to be split. (Why?) Thus the only possibilities are 6 and 2, whose sum is 8, not 10. Hense $x^2 + 10x + 12$ is a polynomial over the integers which cannot be factored into two polynomials over the integers.

4. $a^2 - 9a - 36$. Once again, we wish to write this in the form $(a + m)(a + n)$. What do we know about m and n? What will be the sum of m and n? The product of m and n? Do you see that one will be positive and one will be negative? Which of these will have the larger absolute value?

Now how do we state the relevant problem about factors of 36? Let us consider $|m|$ and $|n|$. Their product must be 36; what does "$m + n = -9$" say about $|m|$ and $|n|$? Recall the definition of addition for one positive and one negative number. In terms of $|m|$ and $|n|$, our problem now is: Find two numbers whose product is 36 and whose difference is 9. What is the prime factorization of 36? Are the two 2's going to be together, or will they be split? What about the two 3's? Therefore, what are $|m|$ and $|n|$? What, finally, is the factorization of $a^2 - 9a - 36$?

5. $3w^2 + 11w$
6. $x^2 + 6sy - 55y^2$
7. $-50s + 48st + 2st^2$
8. $y^4 + 19y^2 + 48$
9. $r^2 - 12(r + 5) + r$
10. $10u - u^2 - 24$
11. $(2 + x)(3 - x)$
12. $a^2 + 2a - 15$
13. $2a^2b - 16aby$
14. $-96n + 42m^2 + 3m^2n$
15. $12 - 7a^2 + a^4$
16. $x^2 - 10(x + 3) - 9$
17. $36 + x^2 - 13x$
18. $2yx^2 - 10y + 6xy$
19. $b^4 + 2b^2 + 3$
20. $2a^2bx^2 - 12a^2by^2 + 2a^2bxy$

21. For which positive integers k can you factor the polynomial

$$x^2 - kx - 12$$

over polynomials with integer coefficients?

22. (Optional) Factor $x^2 + 95x - 5400$. If you really understand factors of numbers, you should be able to do this one without too much labor.

9 - 3. Truth sets of polynomial equations. When an equation can be put in a form in which its left side is a polynomial and the right side is 0, we call it a polynomial equation. The truth set of a polynomial equation in one variable can sometimes be found if we can factor the polynomial.

Example 1. Find the truth set of $x^2 - x - 6 = 0$. Our first job is to factor the polynomial, if possible. Then we get the new form of the equation:

$$(x = 2)(x - 3) = 0.$$

The left side of the equation is now the product of two numbers. What can we say in general about two numbers a and b if $ab = 0$? (Go back to Theorem 7-6 for the answer if you have forgotten.) Thus the truth set of the sentence "$(x + 2)(x - 3) = 0$" is the same as the truth set of the compound sentence

$$x + 2 = 0 \quad \text{or} \quad x - 3 = 0$$

(Why was it so important to write the original equation with 0 on the right side? Could we apply Theorem 7-6 if the equation had 1 on the right side?) Now we can read directly the truth set of this compound sentence as $\{-2, 3\}$. That is, we predict that the numbers -2, 3 each must make the original equation true because they make the compound sentence true. Check each of these numbers in the original equation to verify our prediction. It is customary to say the equation has the two solutions -2 and 3.

Example 2. Find the truth set of $2x^2 = 36x$.

This is also a polynomial equation because it can be written in the form

$$2x^2 - 36x = 0,$$

in which the left side is a polynomial and the right side is 0. How did we obtain this form? Again, factor the polynomial:

$$2x(x - 18) = 0.$$

This equation has the same truth set as the sentence

$$2x = 0 \quad \text{or} \quad x - 18 = 0. \qquad \text{(Why?)}$$

And this sentence has the truth set $\{0, 18\}$. Verify that 0 and 18 are solutions of the original equation.

Pause to reflect: Were you tempted to divide both sides of $2x^2 = 36x$ by $2x$ to obtain $x = 18$? Then you would have found the solution 18, but what happened to the solution 0? Be careful: Until we study this more carefully in Chapter 10, do not multiply or divide the sides of an equation by an expression involving the variable.

Example 3. Find the truth set of

$$x^2 - 4x - 9 = 0$$

It is easy to see that there are no (integer) factors of -9 whose sum is -4; hence, we cannot factor the polynomial. This does not mean that the truth set of the equation is necessarily empty; it means that at present we can't find it.

Exercises 9 - 3.

1. Find the truth set of each of the polynomial equations in x:

    (a) $x^2 - 4x - 5 = 0$ (d) $7x^2 - 42x = 0$

    (b) $x^2 - 13x - 30 = 0$ (e) $39x = 13x^2$

    (c) $18 + 7x - x^2 = 0$ (f) $x^2 - 8x - 153 = 0$

    (g) $3x^2 - 11x + 30 = 2x^2 - 2x = 10$ (Can you arrange to get a polynomial on the left, 0 on the right?)

    (h) $10 + 4x^2 - 30x = 6x + 10$

    (i) $2x^2 = 10x - x^2 + 4x + 72$

    (j) $x^2 - 2ax = 0$ (Remember: this is a polynomial equation in x.)

    (k) $x^2 + x - \frac{3}{4} = 0$ (o) $x^3 - 7x^2 + 12x = 0$

    (l) $x^2 - 8x - 36 = 0$ (p) $ax^2 - bx = 0$

    (m) $x^2 + x - \frac{10}{9} = 0$ (q) $x^2 - ax - bx + ab = 0$

    (n) $2x^2 - 3x = 0$ (r) $x^2 - 4x + ax - 4a = 0.$

2. Translate the following into open sentences and find their truth sets:

    (a) The square of a number is 7 greater than 6 times the number. What is the number?

    (b) The sum of two numbers is 15 and the sum of their squares is 137. Find the numbers.

(c) The length of a rectangle is 5 inches more than its width. Its area is 84 square inches. Find its width.

(d) The square of a number is 9 less than 10 times the number. What is the number?

(e) The length of a rectangle is 7 inches more than its width and its diagonal is 15 inches. Find its width.

(f) One number is 8 less than another, and their product is 84. Find the number.

(g) The product of two consecutive odd numbers is 15 more than 4 times the smaller number. What are the numbers?

(h) Starting from the same point, Jim walked north at a certain constant rate, while Bill walked west at a constant rate which was 1 m.p.h. greater than that of Jim. If they were 5 miles apart at the end of 1 hour, what was the walking rate of each?

(i) The altitude of a triangle is 3 inches less than its base. Its area is 14 square inches. What is the length of its base?

(j) Find the dimensions of a rectangle whose perimeter 28 feet and whose area is 24 square feet.

9 - 4. Perfect squares. Factor $y^2 + 2y + 9$. This time we require two factors of 9 whose sum is 2. Can you find such factors? Do you think you can find two factors of 9 whose sum is 4? What is the smallest positive number which is the sum of these two factors?

Now try the same thing with $x^2 + kx + 36$? How small a positive number do you think k can be? What about $t^2 + kt + 4$? $s^2 + ks + 64$? $u^2 + ku + 100$? Do you see the pattern? Apparently if you look for the smallest positive k such that

$$x^2 + kx + n^2$$

can be factored, where n is a positive integer, it turns out that $k = 2n$. Another way of saying this, a way which uses our knowledge of factors, is that if two positive integers have a product equal to n^2, then their sum must be at least $2n$. But we observe something else: if $k = 2n$, the two factors of n^2 are both equal to n. In other words,

$$x^2 + 2nx + n^2 = (x + n)(x + n) = (x + n)^2.$$

Our polynomial has turned out to be a perfect square.

Exercises 9 - 4.

1. What is the greatest negative number k for which

$$x^2 + kx + n^2$$

can be factored? If k is this greatest negative number, what is the factored form of the polynomial? How should you recognize whether or not

$$x^2 + ax + b$$

is a perfect square in general?

2. If any of the following polynomials is a perfect square, write it in factored form. If not, state why not.

(a) $t^2 + 18t + 81$

(b) $t^2 - 18t - 81$

(c) $4x + 4$

(d) $y^2 + 10x + 25$

(e) $s^2 - 10s - 24$

(f) $a^2 - 2ab^2 + b^4$

3. The polynomial

$$x^2 + 5x + 12$$

cannot be factored over the real numbers. Can you tell why? Even though 12 is not a perfect square, does our preceding argument permit you to say how small k can be if

$$x^2 + kx + 12$$

is to be factorable? Between what two integers is $2\sqrt{12}$?

4. Can

$$t^2 - 5t + 24$$

be factored? Can $t^2 - 5t - 24$?

5. Can you factor $x^2 + 6x + 4$? Can you factor $y^2 + 15y + 12$? Can you factor $t^2 - 21t + 20$? What do you suppose the largest k is for which you can factor $x^2 + kx + n$, if n is a given positive integer? What is the smallest negative k?

6. Find the value of k such that each of the following polynomials is a perfect square: (k can be a real number)

(a) $x^2 + kx + 9$

(b) $x^2 - kx + 16$

(c) $x^2 + 4x + k$

(d) $x^2 - \frac{4}{3}x + k$

(e) $x^2 + kx + 15$

(f) $x^2 + \sqrt{40}x - k$

7. Examine the following - are they perfect squares?

(a) $a^2 + 2ab + b^2$

(b) $x^2 + 2ax + 9a^2$

(c) $7^2 + 2(7)(5) = 5^2$

(d) $7 + 2\sqrt{7}\sqrt{5} + 5$

(e) $4 - 2\sqrt{15} + \frac{15}{4}$

(f) $x^2 - \frac{x}{2} + \frac{1}{4}$

(g) $(x - 1)^2 - 6(x - 1) + 9$ (Think of this as a polynomial in $(x - 1)$.)

(h) $(2a + 1)^2 + 10(2a + 1) + 25$

8. Factor, if possible: (if not possible, explain why.)

(a) $x^2 - 4x + 60$

(b) $7x^2 + 14x + 7$ (Don't overlook the common monomial factor.)

(c) $\frac{1}{4}x^2 - x + 1$

(d) $\frac{1}{2}ax^2 - a + \frac{17a}{2}$

(e) $x^2 - 4$ (What is the coefficient of x ?)

(f) $x^2 + 4$

(g) $(x + 2)^2 - 4(x + 2) + 60$

9. Write the squares:

(a) $(x + 3)^2 =$

(b) $(x - 2)^2 =$

(c) $(x + \sqrt{2})^2 =$

(d) $(a + b)^2 =$

(e) $(x - y)^2 =$

(f) $(x - 1)^2 - a^2 = ?$

(g) $\big((x - 1) + a\big)\big((x - 1) - a\big) = ?$

10. Find the truth set of each of the polynomial equations:

(a) $y^2 - 10y - 24 = 0$

(b) $\frac{1}{2}x^2 - 2x + 2 = 0$ (Hint: multiply both sides by 2.)

(c) $(x - 1)^2 - 4 = 0$

9 - 5. Difference of squares. Can you factor $x^2 + 9$? What is the coefficient of the linear term here? Is it large enough so that the polynomial is factorable?

Now let us try $x^2 - 9$. If we wish to write this polynomial as $(x + p)(x + q)$, what are the conditions on p and q? Can you find two numbers whose product is -9 and whose sum is 0? What does it mean for two numbers to have the sum 0? Check the factorization you have obtained by multiplying the factors. What happened to the linear term?

Exercises 9 - 5a.

1. Factor $y^2 - 36$.
2. Factor $2s^2 - 50$.
3. Factor $at^2 - 16a$.
4. Factor $4y^2$.
5. Factor $25x^2 - 36$. Under what conditions does this particularly simple case of factoring seem to work out? State a general condition for such factoring. Can you factor

$$a^2 - b^6$$

in the same way? Is b^6 the square of anything?

Now suppose you are asked to factor $t^2 - 6$. If you think about this a while, you will find yourself in a dilemma. Certainly you know a number whose square is 6, namely $\sqrt{6}$; hence, it is without doubt true that

$$t^2 - 6 = (t - \sqrt{6})(t + \sqrt{6}).$$

But now you are unhappy because $t^2 - 6$ is a polynomial over the integers, whereas the two factors we have found are not polynomials over the integers. (Why not?) Up to now we have always insisted that we wanted to factor polynomials over the integers into factors which were the same kind of polynomials. Are we changing our mind?

You see that we have to be careful to say explicitly over what set of polynomials we wish to factor a given polynomial. It will turn out later in this course that many polynomials which we have not been able to factor into two polynomials over the integers can, in fact, be factored into polynomials over the real numbers.

Consider $x^2 + 9$ again. We can write this as $x^2 - (-9)$; why can we not factor this as the difference of two squares? Do you know any number whose square is -9? This suggests that there are polynomials that cannot be factored into the product of polynomials over the real numbers. More of this in another year and another course.

Exercises 9 - 5b.

1. Factor the following over the integers, if possible; if not, then over the real numbers, if possible. In each case state whether you are factoring over the integers, or over the real numbers, or whether the polynomial is not factorable.

(a) $r^2 - 18$ (d) $2m^2 - 288$

(b) $x^2 - 5$ (e) $bx^2 - 25b$

(c) $x^2 + 4$ (f) $3xa^2 - 15xb^2$

(g) $a^4 - b^6$

(h) $8 - 2a^{10}$

(i) $4x^2y^2 - 9a^2$

(j) $27x^2y^2 - 48x^2r^2$

(k) $16y^2 - 225r^6$

(l) $2r^2 - 14$

(m) $x^2y^4 - 9a^4b^6$

(n) $ab^4 + 4ay^4$

(o) $(x - 1)^2 - 1$

(p) $(3 - a)^2 - (a + 1)^2$

(q) $(a + 2)^2 + 1$

2. Factor $20^2 - 1$. Solution:

$$\begin{aligned} 20^2 - 1 &= 20^2 - 1^2 && \text{(Why?)} \\ &= (20 - 1)(20 + 1) && \text{(Why?)} \\ &= 19 \cdot 21 . \end{aligned}$$

Can you see how to reverse these steps? Suppose you are asked to find (19)(21) mentally? Is it easier to find $20^2 - 1$?

Find mentally:

(a) (22)(18)

(b) (37)(43)

(c) $(26r)(34)$

(d) $(23x)(17y)$

(e) (101)(99)

(f) $(40m)(50n)$

(g) $36(m + n)\big(44(m - n)\big)$

(h) (6)(6)(4)(11)

3. Can 899 be a prime number? (Hint: $899 = 30^2 - 1$.)

4. Can 1591 be a prime number?

5. Can you tell anything about the factors of 391?

6. Can you tell anything about the factors of 401?

7. Find the truth set of:

(a) $9y^2 - 36 = 0$

(b) $2x^2 = 32$

(c) $(x - 3)^2 = 4$

8. (Optional) What is $(2 - \sqrt{3})(2 + \sqrt{3})$? Once again, since we have the sum and difference of the same two numbers, this becomes $(2)^2 - (\sqrt{3})^2 = 4 - 3 = 1$. We can apply this to rationalize the denominator in

$$\frac{1}{2 - \sqrt{3}}.$$

By the multiplication property of 1,

$$\frac{1}{2 - \sqrt{3}} = \frac{1}{2 - \sqrt{3}} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$

$$= \frac{2 + \sqrt{3}}{1} = 2 + \sqrt{3}.$$

What does this say about the reciprocal of $2 - \sqrt{3}$? Of $2 + \sqrt{3}$?

Rationalize the denominator:

(a) $\frac{2}{5 + \sqrt{2}}$ (b) $\frac{3 + \sqrt{5}}{3 - \sqrt{5}}$ (c) $\frac{-6}{2 + \sqrt{4}}$ (d) $\frac{\sqrt{3}}{\sqrt{6} - \sqrt{5}}$.

9 - 6. Quadratic polynomials. We had no trouble factoring the quadratic polynomial $25x^2 - 36$ in the preceding exercises; we found that

$$25x^2 - 36 = (5x - 6)(5x + 6).$$

Now suppose we try the quadratic polynomial

$$25x^2 - 45x - 36.$$

Can we factor this? How does it differ from most of our previous examples? Before we try this one, let us see what happens when we perform multiplications like

(a) $(2x + 3)(3x + 5)$, (b) $(3y + 7)(4y - 3)$,

(c) $(ax + b)(cx + d)$.

How did the quadratic term in the distributed form for each of these products arise? How did the constant term arise? What did you do to get the coefficient of x in the linear term in Example (a)? Or of y in Example (b)? Does the following diagram suggest how the linear term arose in Example (c) ?

$$(ax + b)(cx + d)$$

The coefficient of the linear term is the sum of ad and bc. Let us use these ideas to factor some polynomials.

Example 1: Factor $2x^2 + 11x + 15$.

We must find factors of 2, $ac = 2$; factors of 15, $bd = 15$; and the sum of ad and b must be 11. Now 2 can be factored in only one way, since it is a prime. So we write

$$(2x + b)(x + d).$$

15 has two factorizations: $15 \cdot 1$ or $5 \cdot 3$. Since we must add the outside and inside products, $15 \cdot 1$ is unsuitable, since neither of the sums $15 + 2$ or $30 + 1$ is 11. So the two possibilities are

$$(2x + 3)(x + 5) \quad \text{and} \quad (2x + 5)(x + 3) .$$

The first of these has a linear term $13x$ and the second, the linear term $11x$. Thus

$$2x^2 + 11x + 15 = (2x + 5)(x + 3) .$$

We used our knowledge of factors here, but the theorems of Chapter 8 were not necessary because 2 and 15 have no common proper factors.

Example 2: Factor $6x^2 + 7x - 24$.

There are a number of possible ways of factoring 6 and 24. Simply trying all possible factorizations as we did in Example 1 could be time consuming. But prime factorization and theorems 8 - 2 and 8 - 3 can help here. First, we note that

$$6 = 2 \cdot 3 , \quad 24 = 2 \cdot 2 \cdot 2 \cdot 3 .$$

Recall the form

$$(ax + b)(cx + d) = acx + (ad + bc)x + bc.$$

We want to form ad and bc such that their sum is 7. The products ad and bc must be made up of the four 2's and two 3's which make up 6 and 24. Since 2 is not a factor of 7, all the 2's must go together (Theorem 8 - 2). Since 3 is not a factor of 7, all the 3's must go together. Therefore we have exactly two possibilities.

ad	bc	ad	bc
$2\cdot2\cdot2\cdot2\cdot3\cdot3$	1	$2\cdot2\cdot2\cdot2$	$3\cdot3$

The polynomial can be written $6x^2 + 7x + (-24)$, and, since the constant term is negative, we know that 7 is the difference, $|ad - bc|$. The difference in the first possibility is obviously too large; so if the polynomial is factorable, it must be the second possibility. And $|16 - 9|$ is 7. Now all that remains is to select a from ad and c from bc so that $ac = 6$. We take $a = 2$ and $c = 3$;

then we have:

$$(2x \quad 3)(3x \quad 2\cdot 2\cdot 2)$$

The larger of ad and bc is positive; so

$$6x^2 + 7x - 24 = (2x - 3)(3x + 8) .$$

Example 3: Factor $25x^2 - 45x - 36$.

$25 = 5^2$, $36 = 2^2 \cdot 3^2$.

Since 5 is a factor of 45, the 5's are split. (Theorem 8 - 3)

Since 3 is a factor of 45, the 3's are split. (Theorem 8 - 3)

Since 2 is not a factor of 45, the 2's must be together.

Therefore, the outside and inside products are

$$5 \cdot 3 \cdot 2 \cdot 2 \quad \text{and} \quad 5 \cdot 3 .$$

We select a 5 from each product to form 25; thus

$$(5x \quad 3)(5x \quad 3\cdot 2^2) .$$

Finally, $25x^2 - 45x - 36 = (5x + 3)(5x - 12)$.

Example 4: Factor $6x^2 + 143x - 24$.

$6 = 2 \cdot 3$, $24 = 2^3 \cdot 3$.

2 is not a factor of 143; so the 2's must be together.

3 is not a factor of 143, so the 3's must be together.

Since 143 is very large, we suspect all the 2's and all the 3's belong in the same product. So the outside and inside products are

$$2^4 \cdot 3^2 \quad \text{and} \quad 1 .$$

We select $2 \cdot 3$ from one product and 1 from the other:

$$(2\cdot 3x \quad 1)(x \quad 2^3\cdot 3)$$

Therefore,

$$6x^2 + 143x - 24 = (6x - 1)(x + 24) .$$

The fact that 143 is so large compared with 6 and 24 may suggest trying the above factors before prime factorization is done.

If the middle term of Example 3 had been $-15x$, would the argument have been any different? Can you factor $25x^2 - 15x - 36$ into two polynomials over the integers?

Exercises 9 - 6a.

1. Factor, if possible, into polynomials over the integers:

(a) $2x^2 + x - 3$

(b) $x^2 + 4x - 32$

(c) $4y^2 + 23y - 6$

(d) $8a^2 + 10a - 3$

(e) $3c^2 - 2c - 6$

(f) $9x^2 - 4$

(g) $9a^2 + 3a - 2$

(h) $9x^2 - 25x - 6$

(i) $6 - 23a - 4a^2$

(j) $15b^2 - 4b - 4$

(k) $7x^2 + 19x - 6$

(l) $p^2 - 13pq - 48q^2$

(m) $3x^2 - 17x - 6$

(n) $12a^2 + 5x - 27$

(o) $12x^2 + 19xy - 10y^2$

(p) $14a^2 + 61a - 9$

(q) $10x^2 + 43x + 45$

(r) $10x^2 - 69x - 45$

(s) $2x^2 - 19x - 24$

(t) $4x^2 - 15x + 14$

(u) $4x^2 - 24x + 36$

(v) $3(x - 1)^2 - 22(x - 1) - 16$

(w) $4(2 - a)^2 - 7(2 - a) + 12$

2. Factor:

   (a) $6x^2 - 144x - 150$
   (b) $6x^2 - 11x - 150$
   (c) $6x^2 + 60x + 150$
   (d) $6x^2 - 61x + 150$
   (e) $6x^2 + 25x + 150$
   (f) $6x^2 + 65x + 150$
   (g) $6x^2 - 87x + 150$
   (h) $6x^2 + 63x - 150$

3. Can $2x^2 + ax + b$ be factored if a is even and b is odd? Why?

4. Can $3x^2 + 5x + b$ be factored if 3 is a factor of b? If so choose a value of b such that $3x^2 + 5x + b$ can be factored.

5. Find the truth set of:

   (a) $8x^2 + 10x - 3 = 0$
   (b) $6y^2 + y = 1$
   (c) $6v^2 = 19v + 7$
   (d) $2x^2 + 9bx + 4b^2 = 0$, for all real b .
   (e) $9x^2 = 4x$.
   (f) $9x^2 = 4$.
   (g) $9(x - 1)^2 = 4$.
   (h) $9(x - a)^2 = 4$, for all real a .

To summarize, if a quadratic polynomial can be factored by inspection into two polynomials over the real numbers, then one of the following forms applies:

$$a^2 - b^2 = (a + b)(a - b) \qquad \text{(Say this in words)}$$

$$a^2 + 2ab + b^2 = (a + b)^2 \qquad \text{(Say this in words)}$$

$$ax^2 + bx + c = (mx + n)(px + q),$$

where $mp = a$, $nq = c$ and $mq + np = b$.

Exercises 9 - 6b.

Factor:

1. $(v + 1)^2 - 16$

2. $(4x^2 - 4x + 1) - y^2$ (Do you recognize $4x^2 - 4x + 1$ as a perfect square?)
3. $(2u - 1)^2 + 5(2u - 1)$
4. $(2u - 1)^2 + 5(2u - 1) + 4$
5. $x^2 + 2x(y - 1) + (y - 1)^2$
6. $(a + b + c)^2 - 4$
7. $6r^2s^2t^2 - 24r^2s^2(t - 4)^2$
8. $6(3c + 1)^2 - 17(3c + 1) - 3$
9. $(9d^2 - 6d + 1) - 4(d^2 + 8d + 16)$
10. $x^4 + 4$ (Hint: $x^4 + 4 = (x^4 + 4x^2 + 4) - 4x^2$. Why?)

9 - 7. Rational expressions. Let us recall some meanings of words in mathematics. An integer is either a counting number of arithmetic, 0, or the opposite of a counting number. The set of integers is closed under addition, subtraction and multiplication, but not under division. What is a polynomial? (Sometimes it is called an integral expression. Can you guess why?)

A rational number is the quotient of two integers $\frac{a}{b}$, with $b \neq 0$. We learned that the set of rational numbers is closed under addition, subtraction, multiplication, and division except by 0. Does this suggest what we shall mean by a rational expression?

Thus, by a rational expression in one variable we mean an expression obtained by performing only the four operations on real numbers, some of which may be represented by the variable.

For example,

$$\frac{1}{x} + 1, \quad \frac{2x - 3}{4x^2 - 9}, \quad \frac{\sqrt{2x}^2 + 5}{3x}, \quad \frac{2}{2x} + \frac{3}{x - 1}, \quad 3x - 2$$

are rational expressions in one variable x. Some of these (which ones?) are quotients of polynomials (integral expressions), just as rational numbers are quotients of integers. Some of these (which ones?) are polynomials (integral expressions), just as some rational numbers are integers. Why is $\sqrt{2x-1}$ not a rational expression?

Now we are ready to perform operations on rational expressions. Remember that for each value of the variable, a rational expression is a real number (if it is defined). This means that the same properties hold for rational expressions as for real numbers.

It has already been shown that for real numbers a, b, c and d,

(1) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, (Is this true for $b = 0$ or $d = 0$?)

(2) $\frac{ac}{bc} = \frac{a}{b}$, (Is this true for $c = 0$?)

(3) $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$. (Is this true for $b = 0$?)

State each of these properties in your own words. Why can these be applied to rational expressions? What restrictions must be placed on the rational expressions? (What can be said of

$\frac{x - 3}{2x - 4}$ if $x = 2$?) Again there is the question of what shall be the simplest form of the result when we operate on rational expressions. As in the case of rational numbers, we want a form in which there are no polynomial (integral) factors the same in numerator and denominator.

Example 1. $\frac{ax - bx}{x^2} \cdot \frac{a^2 - 2ab + b^2}{a^2 - b^2}$.

To multiply these rational expressions we factor every polynomial and apply the property (1) of multiplication of fractions:

$$\frac{ax - bx}{x^2} \cdot \frac{a^2 + 2ab + b^2}{a^2 - b^2} = \frac{(a - b)(x)(a + b)^2}{x^2\,(a - b)(a + b)},$$

$$= \frac{(a + b)}{x}\,\frac{\big(x(a - b)(a + b)\big)}{\big(x(a - b)(a + b)\big)},$$

by associative and commutative properties.

$$= \frac{a + b}{x},$$

by the multiplication property of 1.

This simplification of the product of two rational expressions was done in a very formal manner. We must remember that the last step would be meaningless if $a = b$ or $a = -b$ or $x = 0$. (Why?) Hence, if $a \neq b$, $a \neq -b$, $x \neq 0$, then

$$\frac{ax - bx}{x^2} \cdot \frac{a^2 - 2ab + b^2}{a^2 - b^2} = \frac{a + b}{x}.$$

Example 2. $$\frac{1 - x^2}{x + 1} \cdot \frac{2 - x}{x^2 - 3x + 2} .$$

Factoring and applying the property of multiplication of fractions, we have

$$\begin{aligned}\frac{1 - x^2}{1 + x} \cdot \frac{x - 2}{x^2 - 3x + 2} &= \frac{(1 - x)(1 + x)(x - 2)}{(1 + x)(x - 2)(x - 1)} , \\ &= \frac{(1 - x)(1 + x)(x - 2)}{(x - 1)\big((1 + x)(x - 2)\big)} , \qquad \text{(Why?)} \\ &= \frac{1 - x}{x - 1}\end{aligned}$$

(But we need not stop here. We should recognize that $1 - x$ and $x - 1$ are opposites.)

$$\begin{aligned}&= \frac{(-1)(x - 1)}{x - 1} , \qquad \text{(Why?)} \\ &= -1 . \qquad \text{(Why?)}\end{aligned}$$

Thus we have shown that if $x \neq -1$, $x \neq 1$, $x \neq 2$, then

$$\frac{1 - x^2}{1 + x} \cdot \frac{x - 2}{x^2 - 3x + 2} = -1.$$

Example 3. $$\frac{\dfrac{x^2 + x - 2}{x^2 - 4x + 4}}{\dfrac{x + 2}{x - 2}}$$

After we factor each polynomial in this quotient of two rational expressions, notice that the multiplication property of 1 will eliminate denominators immediately, just as for quotients of rational numbers.

$$\frac{\dfrac{x^2+x-2}{x^2-4x+4}}{\dfrac{x+2}{x-2}} = \frac{\dfrac{(x+2)(x-1)}{(x-2)(x-2)}}{\dfrac{x+2}{x-2}} \cdot \frac{(x-2)(x-2)}{(x-2)(x-2)},$$

(Why did we write 1 as $\frac{(x-2)(x-2)}{(x-2)(x-2)}$?)

$$= \frac{\dfrac{(x+2)(x-1)(x-2)(x-2)}{(x-2)(x-2)}}{\dfrac{(x+2)(x-2)(x-2)}{(x-2)}}, \qquad \text{(Why?)}$$

$$= \frac{(x+2)(x-1)}{(x+2)(x-2)}, \qquad \text{(Why?)}$$

$$= \frac{x-1}{x-2}. \qquad \text{(Why?)}$$

Hence, if $x \neq 2$, $x \neq -2$, then

$$\frac{\dfrac{x^2+x-2}{x^2-4x+4}}{\dfrac{x+2}{x-2}} = \frac{x-1}{x-2}.$$

Exercises 9 - 7.

Perform the indicated operations and simplify, noting restrictions on the values of the variables:

1. $\frac{3x-3}{x^2-1}$

2. $\frac{x^3-x^2y}{1-y}$

3. $\frac{x^2-4x-12}{x^2-5x-6}$

4. $\frac{ab+ab^2}{a-ab^2} \cdot \frac{1-b}{1+b}$

5. $\frac{\dfrac{x^2-9}{6}}{\dfrac{x^2-3x}{3x+3}}$

6. $\frac{\dfrac{x^2+2x+1}{x^2-1}}{\dfrac{x+1}{x-1}}$

7. $\frac{c - 1}{c} \cdot \frac{c^2}{c^2 - 1}$

8. $\dfrac{\frac{m^2 - 9n^2}{2}}{m = 3n}$

9. $\frac{x^2 - 2x - 3}{x^2 - 4x + 3} \cdot \frac{x^2 - 9}{x^2 - 1}$

10. $\frac{1 - x}{(x - 1)^2}$

11. $\frac{4a^2 - 9b^2}{6a^2 - 9ab} \cdot \frac{6ab}{4ab + 6b^2}$

12. $\frac{x^2 + y^2}{x + y}$

13. $\frac{15x^2 - 26x - 21}{6x^2 - 17x + 7}$

14. $\dfrac{\frac{x^2 - y^2}{x^2 + 2xy + y^2}}{\frac{xz - yz}{xz + yz}}$

15. $\frac{x^2 - 3x + 2}{4x^2 - 16} \cdot \frac{2x + 4}{x^2 - 1}$

16. $\frac{1 - x}{x - 2} \cdot \frac{x^2 - 4}{x^2 - 1}$

17. $\frac{12x + 35x + 18}{9x^2 + 12x + 4} \cdot \frac{3x^2 - 2x}{9x^2 - 4}$

18. $\frac{x^2 - x - 2}{x^2} \cdot \frac{x^2 + x - 2}{9x} \cdot \frac{54x^3}{x^4 - 5x^2 + 4}$

19. $\frac{10x^2 - 7x - 45}{2x^2 - 11x + 15} \cdot \frac{2x^2 - 9x + 9}{10x^2 + 3x - 27}$

20. $\dfrac{\frac{2x^2 - 5x - 3}{x^2 - 4x}}{\frac{x^2 - 3x}{8 - 2x}}$

21. $\dfrac{\frac{x^2 - 1}{x^2 + 1}}{\frac{x^2 - 2x + 1}{x + 1}}$

22. $\frac{4x^2 - 12x + 9 - y^2}{2x + y} \cdot \frac{2x^2 + xy}{2x + y - 3}$

23. $(\frac{1}{2} - \frac{y}{x}) \cdot (\frac{x^2}{x - 2y})$

24. Consider the set of all values of rational expressions for real values of the variables. Do you think this set is closed under each of the four operations of arithmetic?

9 - 8. Addition of rational expressions. The problem of finding the least common denominator of two rational numbers was solved in Chapter 8. We found the prime factor representation of each denominator and then formed the least common denominator (L.C.D.) by collecting the smallest set of factors from the denominators in such a way that each denominator was a factor of the L.C.D. In precisely the same way we form the L.C.D. of rational expressions as a set of polynomial factors of the denominators. Then it is an easy step to addition of the rational expressions. When in doubt, ask yourself, "How did I do it with rational numbers?"

Example 1. $$\frac{7}{36a^5b^2c^3} + \frac{5}{24a^3b^5c^4} .$$

The factored forms of the denominators are:

$$36a^5b^2c^3 = 2^2 3^2 a^5 b^2 c^3 ,$$

$$24a^3b^5c^4 = 2^3 3 a^3 b^5 c^4 .$$

Choosing each factor the greatest number of times it occurs in either denominator, the L.C.D. is $2^3 3^2 a^5 b^5 c^4$. Then, by the multiplication property of 1, we may supply the missing factors by writing:

$$\frac{7}{36a^5b^2c^3} + \frac{5}{24a^2b^5c^4} = \frac{7}{2^2 3^2 a^5 b^2 c^3} \cdot \frac{2b^3c}{2b^3c} + \frac{5}{2^3 3 a^3 b^5 c^4} \cdot \frac{3a^2}{3a^2}$$

$$= \frac{14b^3c}{2^3 3^2 a^5 b^5 c} + \frac{15a^2}{2^3 3^2 a^5 b^5 c^4} \qquad \text{(Why?)}$$

$$= \frac{14b^3c + 15a^2}{72a^5b^5c^4} \qquad \text{(Why?)}$$

Example 2. $$\frac{3}{12 - x - x^2} + \frac{2}{x^2 + 8x + 16} .$$

Since $12-x-x^2 = (4+x)(3-x) = (-1)(x+4)(x-3)$ (Why?)

and $x^2 - 8x + 15 = (x - 3)(x - 5)$, the L.C.D. is

$(-1)(x - 3)(x + 4)(x - 5)$. Now

$$\frac{3}{12-x-x^2} + \frac{2}{x^2-8x+15} = \frac{3}{(-1)(x+4)(x-3)} \cdot \frac{x-5}{x-5} + \frac{2}{(x-3)(x-5)} \cdot \frac{(-1)(x+4)}{(-1)(x+4)}$$

(What restrictions on x?)

$$= \frac{3x - 15}{(-1)(x-3)(x+4)(x-5)} + \frac{-2x - 8}{(-1)(x-3)(x+4)(x-5)} \quad \text{(Why?)}$$

$$= \frac{3x - 15 - 2x - 8}{(-1)(x - 3)(x + 4)(x - 5)} \quad \text{(Why?)}$$

$$= \frac{x - 23}{(3 - x)(x + 4)(x - 5)} . \quad \text{(Why?)}$$

Example 3. $$(1 - \frac{1}{x + 1}) \cdot (1 + \frac{1}{x - 1}).$$

Since $1 - \frac{1}{x + 1} = 1 \cdot \frac{x + 1}{x + 1} - \frac{1}{x + 1}$

and $1 + \frac{1}{x - 1} = 1 \cdot \frac{x - 1}{x - 1} + \frac{1}{x - 1} = \frac{(x - 1) + 1}{x - 1} = \frac{x}{x - 1}$,

then $(1 - \frac{1}{x + 1}) \cdot (1 + \frac{1}{x - 1}) = \frac{x}{x + 1} \cdot \frac{x}{x - 1} = \frac{x^2}{x^2 - 1}$.

Explain all the steps and the restrictions on the value of x.

Exercises 9 - 8.

Perform the following operations, simplify, and keep a record of the restrictions on the values of the variables:

1. $\frac{x - 1}{2} + \frac{x - 3}{4}$

2. $\frac{3}{x^2} - \frac{2}{5x}$

3. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

4. $\frac{5}{x - 1} + 1$

5. $\frac{1}{a^2} - \frac{1}{2a} - 2$

6. $\frac{x}{x + y} - \frac{y}{x - y}$

7. $\frac{4}{m - n} + \frac{5}{n}$

8. $\frac{3}{m - 1} - \frac{2}{m - 2}$

9. $\frac{2}{a - b} + \frac{3}{a - b}$

10. $\frac{x}{x + 5} + \frac{x}{x - 3}$

11. $\frac{5x}{x^2 - 9} + \frac{7}{x + 3}$

12. $\frac{y}{x - a} - 4$

13. $\frac{2a}{(a - b)^2} + \frac{3}{a - b}$

14. $\frac{6}{a^2 - 2ab + b^2} + \frac{7}{a - b}$

15. $\frac{4}{x^2 - y^2} - \frac{5}{x + y}$

16. $\frac{4}{x^2 - x} + \frac{3}{x - 1} - \frac{1}{x}$

17. $\frac{5}{x^2 + x - 6} + \frac{3}{x^2 - 4x + 4}$

18. $\frac{2x}{x^2 - 36} - \frac{4(x - 6)}{x - 6}$

19. $\frac{3}{x^2 + x - 2} - \frac{5}{x^2 - 4x + 3}$

20. $\left(2 + \frac{3x - 3}{x^2 - 4x}\right) \cdot \left(\frac{x^2 - 3x}{8 - 2x}\right)$

21. $\dfrac{x - \frac{y^2}{x}}{1 + \frac{y}{x}}$

22. $\frac{3y}{2y^2 - 8} + \frac{2}{2 - y}$

23. $\frac{a}{a^2 - 10a + 25 - b^2} + \frac{1}{a + b - 5}$

24. Compare the properties of rational numbers and the properties of rational expressions. Can you make any general comparison?

9 - 9. Division of polynomials. Suppose, in the course of simplifying rational expressions you come upon this example:

$$\frac{x^3 + 3x^2 - 38x - 10}{x - 5}$$

You would probably try to factor the polynomial in the numerator, hoping to make use of the multiplication property of 1 to simplify the expression. You would very likely find this polynomial difficult to factor, however, and then you might wonder: Can this expression be simplified at all, and if so, how?

Reading the expression itself gives a clue:

$$x^3 + 3x^2 - 38x - 10 \quad \underline{\text{divided by}} \quad x - 5.$$

This suggests that a process of division for polynomials might be helpful in simplifying the example.

You may be surprised to find that our old friend, the distributive property, will be the tool for dividing such expressions.

First let us recall some ideas about division of real numbers. We defined $\frac{126}{3}$ to be $126 \cdot \frac{1}{3}$, and we proved a theorem to the effect that $\frac{126}{3}$ is that number which multiplied by 3 will give 126.

We also may write

$$\frac{126}{3} = 40 + 2 ,$$

because $3 \cdot 40 + 3 \cdot 2 = 126$.

How can we apply these ideas to

$$\frac{x^3 + 3x^2 - 38x - 10}{x - 5} ?$$

We need to ask for an expression which, multiplied by $(x - 5)$, will be contained in $x^3 - 3x^2 - 10$, just as 40 is a number which, multiplied by 3, is contained in 126. Clearly, to obtain the x^3 which is contained in $x^3 + 3x^2 - 38x - 10$, we must multiply $x - 5$ by x^2. Corresponding, then, to the equation

$$126 = 3\cdot 40 + 3\cdot 2$$

(Notice that this is what you are saying when you take the first step in the "short division" example: $\underline{3}\overset{4}{\overline{)126}}$.) we have the equation

$$x^3 + 3x^2 - 38x - 10 = (x - 5)\,x^2 + ?$$

What do we need to do to complete this equation? Why?

Since $(x - 5)\,x^2 = x^3 - 5x^2$, the subtraction necessary to complete this equation can be written

$$(1)\qquad \begin{array}{r} x^3 + 3x^2 - 38x - 10 \\ \underline{x^3 - 5x^2 } \\ 8x^2 - 38x - 10 \end{array}\ ;$$

so we have

$$(2)\qquad x^3 = 3x^2 - 38x - 10 = (x - 5)\,x^2 + 8x^2 - 38x - 10.$$

We might stop here, except that we suspect that some multiple of $x - 5$ is contained in $8x^2 - 38x - 10$. Returning to our arithmetic example for a moment again, we see that 3 is also contained in the last digit of the dividend, so that our short division problem becomes $3\overset{42}{\overline{)126}}$. Just as there was a multiple of 3 in the last part of our arithmetic problem, so we find a multiple of $x - 5$ in our polynomial problem. Now we need an expression which, multiplied by $x - 5$, will be contained in $8x^2 - 38x - 10$. We see that to obtain $8x^2$, we must multiply $(x - 5)$ by $8x$. Now

$$8x^2 - 38x - 10 = (x - 5)\,8x + ?$$

and we complete the equation by subtracting $8x^2 - 40x$ from $8x^2 - 38x - 10$:

$$(3) \qquad \begin{array}{r} 8x^2 - 38x - 10 \\ \underline{8x^2 - 40x \phantom{{}- 10}} \\ 2x - 10\,; \end{array}$$

so

$$(4) \qquad 8x^2 - 38x - 10 = (x - 5)\,8x + 2x - 10\,.$$

If we substitute into equation (2), we obtain

$$(5) \qquad x^3 + 3x^2 - 9x + 2 = (x - 5)x^2 + (x - 5)8x + 2x - 10.$$

Can we still find a multiple of $(x - 5)$ which is contained in $2x - 10$? What about multiplying $x - 5$ by 2 and continuing as before? Explain the following steps:

$$(6) \qquad \begin{array}{rcr} (x - 5)2 & = & 2x - 10 \\ & & 2x - 10 \\ & & \underline{2x - 10} \\ & & 0\,. \end{array}$$

$$(7) \qquad 2x - 10 = (x - 5)2 + 0\,.$$

$$(8) \qquad x^3 + 3x^2 - 9x + 2 = (x - 5)x^2 + (x - 5)8x + (x - 5)2\,.$$

If you will now examine the right side of equation (8) you will see that the expression can be simplified by the distributive property.

$$(9) \qquad x^3 + 3x^2 - 38x - 10 = (x - 5)(x^2 + 8x + 2)\,.$$

Comparison with the equation $126 = 3 \cdot 42$ leads us to conclude that dividing $x^3 + 3x^2 - 38x - 10$ by $x - 5$ gives the quotient

$x^2 + 8x + 2$, that is

$$\frac{x^3 + 3x^2 - 38x - 10}{x - 5} = x^2 + 8x + 2 .$$

You were taught to "check" long division examples in arithmetic. How would you "check" this example? Is $x - 5$ a factor of $x^3 + 3x^2 - 38x - 10$? Is $x^2 + 8x + 2$ a factor of $x^3 + 3x^2 - 38x - 10$?

Let us write the example in the form which shows the use of the multiplication property of 1:

$$\begin{aligned}\frac{x^3 + 3x^2 - 38x - 10}{x - 5} &= \frac{(x^2 + 8x + 2)(x - 5)}{(x - 5)} \\ &= x^2 + 8x + 2 .\end{aligned}$$

Thus you can see that polynomial division may sometimes be an aid in factoring.

The example of polynomial division above came out without a remainder. More often than not, division of polynomials, just as division in arithmetic, does not "come out even"; that is, as we came to expect a quotient and remainder in arithmetic, we may expect the same sort of thing with the division of polynomials. In the example below you should be able to explain all steps except the last; after you have gone over the example and are satisfied that you understand the early steps, we will explain the last step.

Example: $$\frac{6x^3 + 23x^2 + 9x - 19}{3x^2 + x - 2} .$$

Explain each step in the following:

$$(3x^2 + x - 2)2x = 6x^3 + 2x^2 - 4x$$

$$\begin{array}{r} 6x^3 + 23x^2 + 9x - 19 \\ \underline{6x^3 + 2x^2 - 4x \phantom{{}-19}} \\ 21x^2 + 13x - 19 \end{array}$$

$$6x^3 + 23x^2 + 9x - 19 = (3x^2 + x - 2)2x + 21x^2 + 13x - 19$$

$$(3x^2 + x - 2)7 = 21x^2 + 7x - 14$$

$$\begin{array}{r} 21x^2 + 13x - 19 \\ \underline{21x^2 + 7x - 14} \\ 6x - 5 \end{array}$$

$$6x^3 + 23x^2 + 9x - 19 = (3x^2 + x - 2)2x + (3x^2 + x - 2)7 + 6x - 5$$

$$6x^3 + 23x^2 + 9x - 19 = (3x^2 + x - 2)(2x + 7) + 6x - 5$$

Thus,

$$\frac{6x^3 + 23x^2 + 9x - 19}{3x^2 + x - 2} = 2x + 7 + \frac{6x - 5}{3x^2 + x - 2} .$$

To understand the last step of this problem better, let us return to arithmetic for a moment. When we perform the division $\frac{98}{27}$ we find by the long division process, $\begin{array}{r} 3 \\ 27\overline{)\,98} \\ \underline{81} \\ 17 \end{array}$, that $\frac{98}{27} = 3 + \frac{17}{27}$. We say that the quotient is 3 with a remainder of 17. This example, in terms of our present discussion, would appear as

$$\begin{aligned} 98 &= 27 \cdot 3 + 17 \\ &= 27 \cdot 3 + 27 \cdot \tfrac{17}{27} \qquad \text{Why?} \\ &= 27(3 + \tfrac{17}{27}) \qquad \text{Why?} \end{aligned}$$

In our polynomial example, now, we may proceed in the same fashion, starting from the next to the last step:

$$\begin{aligned} 6x^3 + 23x^2 + 9x - 19 &= (3x^2 + x - 2)(2x + 7) + 6x - 5 \\ &= (3x^2 + x - 2)(2x + 7) \\ &\qquad + (3x^2 + x \cdot 2)\left(\frac{6x - 5}{3x^2 + x - 2}\right) \quad \text{(Why?)} \\ &= (3x^2 + x - 2)\left(2x + 7 + \frac{6x - 5}{3x^2 + x - 2}\right). \quad \text{(Why?)} \end{aligned}$$

Just as in arithmetic, our remainder is the numerator of a fraction whose denominator is the divisor.

One further point about the remainder is important. In arithmetic we were always careful to carry out our division until the remainder was less than the divisor. When dividing polynomials we shall continue the process until the degree of the remainder is less than the degree of the divisor. By degree of a polynomial in one variable, we mean the largest exponent of the variable in the polynomial. Our previous example, when written in a compact form, has this appearance:

$$\begin{array}{l|rrrr|ll} \text{Divisor} & & \text{Dividend} & & & \text{Quotient} & \text{Remainder} \\ & & & & & & \\ \underline{3x^2 + x - 2} & 6x^3 & + 23x^2 & + 9x & - 19 & 2x + 7 + & \dfrac{6x - 5}{3x^2 + x - 2} \\ & \underline{6x^3} & \underline{+ \ 2x^2} & \underline{- 4x} & & & \\ & & 21x^2 & + 13x & - 19 & & \\ & & \underline{21x^2} & \underline{+ \ 7x} & \underline{- 14} & & \\ & & & 6x & - \ 5 & & \end{array}$$

(Notice that the successive subtractions are beneath the dividend.)

There we stop, since the divisor, $3x^2 + x - 2$, is a polynomial of second degree, and the remainder, $6x - 5$, is a polynomial of first degree.

Exercises 9 - 9.

Do the following divisions of polynomials, writing your work in the compact form shown above. Be sure that the degree of the remainder is less than the degree of the divisor. For problem 1, write the work also in the detailed form shown in the example, explaining each step as you proceed. In particular, point out every time you use the distributive property.

1. $\frac{2x^2 - 4x + 3}{x - 2}$

2. $\frac{4x^2 - 4x - 15}{2x + 3}$ (Is division necessary?)

3. $\frac{2x^3 - 5x^2 - 8x + 10}{2x + 3}$

4. $\frac{3x^3 + 7x^2 - 5x + 4}{3x - 1}$

5. $\frac{x^3 + x - 1}{x + 2}$ (Be careful when you perform the first subtraction.)

6. $\frac{x^3 - 3x^2 + 7x - 1}{x^2 - 2x + 1}$

7. $\frac{2x^3 - 2x^2 + 4x + 5}{x^2 - 3}$

8. $\frac{5x^3 + 6}{x^2 + 2}$

9. $\frac{3x^3 + 6x}{x^2 + 2}$

10. $\frac{3x^3 + 7x^2 - 4x + 1}{3x + 5}$ (The quotient and remainder may be polynomials over the rational numbers.)

11. $\frac{3x^4 + 4x^3 - 2x^2 - 7}{x - 4}$

12. $\frac{5x^4 - 3x^3 - x + 4}{5x^2 - 2}$

9 - 10. Review Exercises.

In problems 1 through 3, simplify the given expressions.

1. $2\sqrt{18} + 3\sqrt{12} - \sqrt{\frac{1}{2}} - 6\sqrt{\frac{1}{3}}$

2. $\sqrt{3}\sqrt{6a^4}$

3. $\sqrt{(x + y)^3}$

In problems 4 through 6, multiply the factors and simplify the resulting expression.

4. $2\sqrt{3}\,(2 - \sqrt{6})$

5. $(\sqrt{3} + \sqrt{2})^2$

6. $(\sqrt{x} + 1)(\sqrt{x} - 1)$

In problems 7 through 15, factor the given polynomials over the integers, if possible.

7. $x^2 - 22x - 48$

8. $x^2 - 22x + 48$

9. $3a^3b^5 - 6a^2b^3 + 12a^4b^4$

10. $x^2 - y^2 - 4x - 4y$

11. $(x^2 - y^2) + 2(x - y)^2 - 3(x - y)^3$

12. $6a^2 - 19a + 10$

13. $6a^2 + 11a - 10$

14. $4(x - y)^3 + 8(x - y)^2 - 2(y - x)^2$

15. $x^2 + 2as + a^2 - bx - ba - cx - ca$

16. For what positive integral values of k is the polynomial $x^2 + kx + 12$ factorable over the integers?

17. For what positive integral values of k is the polynomial $x^2 + 6x + k$ factorable over the integers?

In problems 18 through 21, simplify the given phrases.

18. Determine the value of k so that $x^2 - 6\sqrt{3}x + k$ is a perfect square.

19. $$\frac{\dfrac{3x^2y^6}{20a^2b^2}}{\dfrac{7(xy^2)^3}{30(ab^2)^2}}$$

20. $$\frac{3}{35a^2} + \frac{13}{25ab} - \frac{5}{7b^2}$$

21. $$\frac{2}{a^2 - ab} + \frac{3}{b^2 - ab} + \frac{4}{ab}$$

22. $$\frac{x}{x^2 - 9} + \frac{2x - 5}{x^2 - 4x + 3} - \frac{3x}{x^2 + 2x - 3}$$

In problems 23 through 26, divide the given polynomials and check.

23. $$\frac{x^3 - 4x^2 + x + 6}{x - 3}$$

24. $$\frac{3x^4 + 14x^3 - 4x^2 - 11x - 2}{3x + 2}$$

25. $$\frac{x^3 - 1}{x + 1}$$

26. $$\frac{x^5 - 1}{x - 1}$$

In problems 27 through 34, find the truth sets for the given sentences.

27. $\frac{3}{5x} - \frac{3}{4x} = \frac{1}{10}$.

28. $\frac{4}{y - 5} = \frac{5}{y}$.

29. $\frac{x + 2}{x + 1} - \frac{x - 1}{x + 2} = 0$.

30. $\frac{5}{n - 3} - \frac{20}{n^2 - 9} = -1$.

31. $\frac{1}{|x - 3|} = 7$.

32. $4x^2 - 243 = x^2$.

33. $3|x|^2 - 2|x| = 0$.

34. $|x|^2 + |x| = 12$.

In problems 35 through 47, translate the problem into an equation or inequality, and solve the problem by finding the truth set of the equation or inequality.

35. One boy can do his paper route in 30 minutes. His substitute did the route in reverse order and required 45 minutes. Working together, toward each other, how long would it be before they met?

36. A candy store made a 40 lb. mixture of creams selling at \$1.00 per pound and nut centers selling at \$1.40 per pound. If the mixture is to sell at \$1.10 per pound, how many pounds of each kind of candy should be used?

37. A 100 gallon container is tested and found to contain 15 % salt. How much of the 100 gallons should be withdrawn and replaced by pure water to make a 10 % solution?

38. A jet travels 10 times as fast as a passenger train. In one hour the jet will travel 120 miles further than the passenger train will go in 8 hours. What is the rate the jet? The train?

39. Two trains 160 miles apart travel towards each other. One is traveling $\frac{2}{3}$ as fast as the other. What is the rate of each if they meet in 3 hours and 12 minutes?

40. A man makes a trip of 300 miles at an average speed of 30 miles per hour and returns at an average speed of 20 miles per hour. What was his average speed for the entire trip?

41. Generalizing problem 40: a man makes a trip of d miles at an average speed of r miles per hour and returns at an average rate of q miles per hour; what was his average rate for the entire trip?

42. The sum of the reciprocals of two successive integers is $\frac{27}{182}$. What are the numbers?

43. Find the average of $\frac{x + 3}{x}$ and $\frac{x - 3}{x}$.

44. The square of a number is 91 more than 6 times the number. Write a corresponding equation and find its truth set.

45. One automobile travels a distance of 360 miles in 1 hour less than a second going 4 miles per hour slower than the first.

Find the rate of the two automobiles.

46. A rug with area of 24 square yards is placed in a room 14 feet by 20 feet leaving a uniform width around the rug. How wide is the strip around the rug? A sketched diagram of the rug upon the floor may help you represent algebraically the length and width of the rug.

47. One leg of a right triangle is 2 feet more than twice the smaller leg. The hypotenuse is 13 feet. What are the lengths of the legs?

48. Tell which of these numbers are rational:

$$\sqrt[3]{\pi^3}\ ,\ \sqrt{.4}\ ,\ \sqrt[3]{.0008}\ ,\ (\sqrt[3]{-1})(\sqrt{.16})\ .$$

49. (Optional) If a two-digit number of the form $10t + u$ is divided by the sum of its digits, the quotient is 4 and the remainder is 3. Find the numbers for which this is true.

50. (Optional) Simplify $1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}$

51. (Optional) Find the truth set of $|x - 5|^2 \geq 9$.

52. (Optional) At what time between 3 and 4 o'clock will the hands of a clock be together? At what time will they be opposite each other?

53. (Optional) A farmer has \$1000 to buy steers at \$25 and cows at \$26. If you know the number of steers and the number of cows are each positive integers, what is the greatest number of animals he may buy, if he must use the entire \$1000.